

## Torsion functors of local cohomology modules for complexes\*

ZHANG Pinger<sup>1</sup>, MA Yajun<sup>2</sup>✉

1. School of Artificial Intelligence, Gansu University of Political Science and Law, Lanzhou 730070, China

2. School of Mathematics and Physics, Lanzhou Jiaotong University, Lanzhou 730070, China

**Abstract:** We show that the torsion module  $\text{Tor}_j^R(R/\mathfrak{a}, H_{\mathfrak{a}}^i(X))$  is in a Serre subcategory for the bounded below  $R$ -complex  $X$ . In addition, we prove the isomorphism  $\text{Tor}_{s-i}^R(R/\mathfrak{a}, X) \cong \text{Tor}_s^R(R/\mathfrak{a}, H_{\mathfrak{a}}^i(X))$  in some case. As an application, the Betti number of a complex  $X$  in a prime ideal  $\mathfrak{p}$  can be computed by the Betti number of the local cohomology modules of  $X$  in  $\mathfrak{p}$ .

**Key words:** local cohomology; torsion functor; Serre subcategory; Betti number

**CLC number:** O154.2    **Document code:** A    **Article ID:** 2097-0137(2025)04-0109-06

Throughout this paper,  $R$  is a commutative Noetherian ring with non-zero identity. Let  $\mathfrak{a}$  be an ideal of  $R$  and  $M$  a finitely generated  $R$ -module. Grothendieck(1962) conjectured that the module  $\text{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^i(M))$  is finitely generated for all  $i$ , where the  $H_{\mathfrak{a}}^i(M)$  is the  $i$ -th local cohomology of  $M$  with respect to  $\mathfrak{a}$ . Hartshorne(1970) gave a counterexample which shows that this conjecture is false even when  $R$  is regular and he posed a question whether  $\text{Ext}_R^j(R/\mathfrak{a}, H_{\mathfrak{a}}^i(M))$  is finitely generated for all  $i$  and  $j$ . Later, Melkersson proved that  $\text{Ext}_R^i(R/\mathfrak{a}, M)$  is finitely generated for all  $i$  if and only if  $\text{Tor}_i^R(R/\mathfrak{a}, M)$  is finitely generated for all  $i$  in Theorem 2.1 (Melkersson, 2005). This motivates us to pose the following question which is a dual of the Hartshorne's question.

**Question 1** If  $\mathfrak{a}$  is an ideal of  $R$  and  $M$  is a finitely generated  $R$ -module, when is

$$\text{Tor}_j^R(R/\mathfrak{a}, H_{\mathfrak{a}}^i(M))$$

finitely generated for all  $i$  and  $j$ ?

Note that there are some attempts to study  $\text{Tor}_0^R(R/\mathfrak{a}, H_{\mathfrak{a}}^i(M))$  in Asgharzadeh et al. (2010) and  $\text{Tor}_j^R(R/\mathfrak{a}, H_{\mathfrak{a}}^i(M))$  in Khashyarmansh(2007). Recently, Dibaei et al. (2011) showed that, certain situation, the torsion modules  $\text{Tor}_j^R(R/\mathfrak{a}, H_{\mathfrak{a}}^i(M))$  are in a Serre subcategory  $\mathcal{S}$  of the category of  $R$ -modules (i. e. the class of  $R$ -modules which is closed under taking submodules, quotients, and extensions). Subsequently, Rahimpour Nasbi et al. (2017) studied the torsion functors of generalized local cohomology modules in  $\mathcal{S}$  and they presented some upper bounds for the flat dimension and the Betti numbers of generalized local cohomology modules.

In this paper, we present some technical results (Lemma 1 and Theorem 1) which show that, certain situations, the torsion module  $\text{Tor}_j^R(R/\mathfrak{a}, H_{\mathfrak{a}}^i(X))$  is in  $\mathcal{S}$  for the bounded below  $R$ -complex  $X$ . In addition, we

\* Received: 2024-09-10

Accepted: 2024-12-11

Published online: 2025-04-27

Supported by Natural Science Foundation of Gansu Province(23JRRA866);

Higher Education Innovation Fund of Gansu Provincial Department of Education(2025A-132);

University-level Scientific Research and Innovation Project of Gansu University  
of Political Science and Law(GZF2024XQN16);

Youth Foundation of Lanzhou Jiaotong University(2023023)

✉ Corresponding author: MA Yajun(yjma@mail.lzjtu.cn)

ZHANG Pinger(zpe7445@gsu.edu.cn)

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ZR20240278

prove the isomorphism  $\mathrm{Tor}_s^R(R/\mathfrak{a}, X) \cong \mathrm{Tor}_s^R(R/\mathfrak{a}, H_a^i(X))$ . As an application, we show that the Betti number of a complex  $X$  in a prime ideal  $\mathfrak{p}$  can be computed by the Betti number of the local cohomology modules of  $X$  in  $\mathfrak{p}$ .

## 1 Preliminary

We recall some notions and facts which we will need later. See (Tarrío et al, 1997; Foxby et al., 2001; Christensen et al., 2006).

**Complexes.** By an  $R$ -complex  $X$  we mean a sequence of  $R$ -modules

$$\cdots \rightarrow X_{i+1} \xrightarrow{d_{i+1}} X_i \xrightarrow{d_i} X_{i-1} \rightarrow \cdots.$$

The derived category  $D(R)$  of the category of  $R$ -complexes is the category of  $R$ -complexes localized at the class of all quasi-isomorphisms. An  $R$ -complex  $X$  is called bounded below (resp. bounded above) if  $H_i(X) = 0$  for  $i \ll 0$  (resp.  $i \gg 0$ ). The full subcategories consisting of bounded below and bounded above  $R$ -complexes are denoted by  $D^+(R)$  and  $D^-(R)$ . By  $D^f(R)$  we mean the full subcategory of  $D(R)$  consisting of complexes  $X$  with  $H_i(X)$  finitely generated  $R$ -modules for all  $i$ . For an  $R$ -complex  $X \in D(R)$ , the supremum and infimum of  $X$  are defined by  $\sup X = \sup \{i \in \mathbb{Z} \mid H_i(X) \neq 0\}$  and  $\inf X = \inf \{i \in \mathbb{Z} \mid H_i(X) \neq 0\}$ , respectively.

Let  $X$  and  $Y$  be two  $R$ -complexes. For every  $i \in \mathbb{Z}$ ,

$$\mathrm{Tor}_i^R(X, Y) := H_i(X \otimes_R Y).$$

We write  $\mathrm{Spec} R$  for the set of prime ideals of  $R$ , set

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec} R \mid \mathfrak{a} \subseteq \mathfrak{p}\}.$$

The support for  $X \in D(R)$  is defined by uniting the corresponding set for the homology modules as follows:

$$\mathrm{Supp}_R X := \bigcup_{i \in \mathbb{Z}} \mathrm{Supp}_R H_i(X) = \{\mathfrak{p} \in \mathrm{Spec} R \mid X_{\mathfrak{p}} \neq 0\}.$$

**Local cohomology.** Let  $M$  be an  $R$ -module. The  $\mathfrak{a}$ -torsion submodule of  $M$  is

$$\Gamma_{\mathfrak{a}}(M) := \{m \in M \mid \mathfrak{a}^s m = 0 \text{ for some integer } s\}.$$

The association  $M \rightarrow \Gamma_{\mathfrak{a}}(M)$  extends to define a left exact additive functor on the category of  $R$ -complexes, its right derived functor is denoted by  $R\Gamma_{\mathfrak{a}}(-)$ , that can be computed by  $R\Gamma_{\mathfrak{a}}(X) \simeq \Gamma_{\mathfrak{a}}(I)$ , where  $X \xrightarrow{\cong} I$  is a semi-injective resolution of  $X$ . For each  $R$ -complex  $X$  and integer  $i$ , the  $i$ -th local cohomology of  $X$  with support in  $\mathfrak{a}$  is the  $R$ -module

$$H_{\mathfrak{a}}^i(X) := H_{-i}(R\Gamma_{\mathfrak{a}}(X)).$$

## 2 Main results and applications

In this section,  $c$  denotes the arithmetic rank of the ideal  $\mathfrak{a}$ , so that there exist elements  $x_1, \dots, x_c$  of  $R$  such that  $\sqrt{\mathfrak{a}} = (x_1, \dots, x_c)$ , also  $C(X)^\cdot$  denotes the Čech complex of  $R$ -complex  $X$  with respect to  $x_1, \dots, x_c$ . It is well known that the  $i$ -th cohomology module of  $C(X)^\cdot$  is isomorphic to the  $i$ -th local cohomology module  $H_{\mathfrak{a}}^i(X)$  of  $X$  (see Theorem 3.2 (Schenzel, 2003)).

Our results are based on the following lemma. We adopt the notation as in Rotman(2009).

**Lemma 1** Let  $N$  be an  $R$ -module with  $\mathrm{Supp}_R N \subseteq V(\mathfrak{a})$  and  $X \in D^+(R)$ . Then there is a first quadrant spectral sequence

$$E_{p,q}^2 := \mathrm{Tor}_p^R(N, H_{\mathfrak{a}}^{c-q}(X)) \Rightarrow \mathrm{Tor}_{p+q-c}^R(N, X). \quad (1)$$

**Proof** Let  $F_\cdot$  be a free resolution of  $N$  and consider the first quadrant bicomplex  $\mathcal{T} = \{C(F_p \otimes_R X)^{c-q}\}$ .

We denote the total complex of  $\mathcal{T}$  by  $\mathrm{Tot}(\mathcal{T})$ . The first filtration has  $E^2$  term the iterated homology  $H_p H_{p,q}''(\mathcal{T})$ . By Theorem 3.2 (Schenzel, 2003), we have

$$H_{p,q}''(\mathcal{T}) = H^{c-q}(C(F_p \otimes_R X)^\cdot) = H_{\mathfrak{a}}^{c-q}(F_p \otimes_R X) = F_p \otimes_R H_{\mathfrak{a}}^{c-q}(X).$$

Hence

$${}^I E_{p,q}^2 = H_p(F \otimes_R H_a^{c-q}(X)) = \text{Tor}_p^R(N, H_a^{c-q}(X)).$$

On the other hand, the second filtration has  $E^2$  term the iterated homology  $H''_p H'_{q,p}(\mathcal{T})$ . We have

$$H'_{q,p}(\mathcal{T}) = H_q(\mathcal{C}(R)^{c-p} \otimes_R F \otimes_R X) = \mathcal{C}(R)^{c-p} \otimes_R H_q(F \otimes_R X) = \mathcal{C}(\text{Tor}_q^R(N, X))^{c-p}.$$

Thus, again by Theorem 3.2 (Schenzel, 2003), we have

$${}^{II} E_{p,q}^2 = H^{c-p}(\mathcal{C}(\text{Tor}_q^R(N, X))) = H_a^{c-p}(\text{Tor}_q^R(N, X)).$$

Since  $\text{Tor}_q^R(N, X)$  is  $\mathfrak{a}$ -torsion for all  $q$ , we have

$${}^{II} E_{p,q}^2 \cong \begin{cases} \text{Tor}_q^R(N, X) & \text{if } p = c, \\ 0 & \text{if } p \neq c. \end{cases}$$

Therefore this spectral sequence collapses at the  $c$ -th column and so

$$H_{p+q}(\text{Tot}(\mathcal{T})) = {}^{II} E_{c,p+q-c}^2 = \text{Tor}_{p+q-c}^R(N, X)$$

which yields the assertion.

In the following theorem, we find some sufficient conditions for the torsion functors of local cohomology modules of complexes in  $\mathcal{S}$ .

**Theorem 1** Let  $N$  be an  $R$ -module with  $\text{Supp}_R N \subseteq V(\mathfrak{a})$ , and let  $X$  be in  $D^+(R)$  and  $s \geq 0, t \geq \inf X$  such that

- (i)  $\text{Tor}_{s-i}^R(N, X)$  is in  $\mathcal{S}$ .
- (ii)  $\text{Tor}_{s-i+1}^R(N, H_a^{t-i}(X))$  is in  $\mathcal{S}$  for all  $1 \leq i \leq t - \inf X$ .
- (iii)  $\text{Tor}_{s+i-1}^R(N, H_a^{t+i}(X))$  is in  $\mathcal{S}$  for all  $1 \leq i \leq c$ .

Then  $\text{Tor}_s^R(N, H_a^t(X))$  belongs to  $\mathcal{S}$ .

**Proof** We may assume that  $t - \inf X \leq c$ . Set  $u = c - t$  and consider the spectral sequence (1). If  $s = 0$ , there exists a finite filtration

$$0 = U^{-1} \subseteq U^0 \subseteq \dots \subseteq U^{u - \inf X} = \text{Tor}_u^R(N, X)$$

such that  $U^p/U^{p-1} \cong E_{p,u-p}^\infty$  for  $u \geq p + \inf X$ . Let  $r \geq 2$ . Consider the differential

$$E_{r,u-r+1}^r \xrightarrow{d_{r,u-r+1}^r} E_{0,u}^r \xrightarrow{d_{0,u}^r} E_{-r,u+r-1}^r = 0.$$

We obtain the following short exact sequence

$$0 \rightarrow \text{Im } d_{r,u-r+1}^r \rightarrow E_{0,u}^r \rightarrow E_{-r,u+r-1}^r \rightarrow 0.$$

Since  $u - r + 1 \leq u - 1$  and  $E_{r,u-r+1}^r$  is a subquotient of  $E_{r,u-r+1}^2$  and  $E_{r,u-r+1}^2 \in \mathcal{S}$  by the condition (ii), it follows that  $\text{Im } d_{r,u-r+1}^r \in \mathcal{S}$  for  $r \geq 2$ . By the condition (i), we have  $E_{0,u}^r \cong E_{0,u}^\infty \cong U^0/U^{-1} \in \mathcal{S}$  for  $r \geq 0$ . By using the above sequence inductively, one has  $N \otimes H_a^t(X) \cong E_{0,u}^2 \in \mathcal{S}$ . The proof the case  $s = 1$  is similar to that for  $s = 0$ .

Assume  $s \geq 2$ . Consider the following filtration

$$0 = U^{-1} \subseteq U^0 \subseteq \dots \subseteq U^{s+u - \inf X} = \text{Tor}_{s-1}^R(N, X),$$

where  $U^p/U^{p-1} \cong E_{p,s+u-p}^\infty$  for  $s+u \geq p + \inf X$ . Let  $r \geq 2$ . Consider the differential

$$E_{s+r,u-r+1}^r \xrightarrow{d_{s+r,u-r+1}^r} E_{s,u}^r \xrightarrow{d_{s,u}^r} E_{s-r,u+r-1}^r.$$

Since  $E_{s-r,u+r-1}^r = 0$  for  $r \geq s + 1$  and  $E_{s+r,u-r+1}^r = 0$  for  $r \geq u + \inf X + 2$ , it follows from the conditions that  $\text{Im } d_{s+r,u-r+1}^r$  and  $\text{Im } d_{s,u}^r$  are in  $\mathcal{S}$  for  $r \geq 2$ . Let  $r \geq s + 1$ . Then  $E_{s-r,u+r-1}^r = 0$ . So we have a short exact sequence

$$0 \rightarrow \text{Im } d_{s+r,u-r+1}^r \rightarrow E_{s,u}^r \rightarrow E_{s,u}^{r+1} \rightarrow 0.$$

Since  $E_{s,u}^r \cong E_{s,u}^\infty \cong U^s/U^{s-1} \in \mathcal{S}$  for  $r \geq 0$ , it follows from that the above sequence that  $E_{s,u}^{s+1} \in \mathcal{S}$ . Thus the following exact sequence

$$0 \rightarrow \text{Im } d_{2s,u-r+1}^s \rightarrow \text{Ker } d_{s,u}^s \rightarrow E_{s,u}^{s+1} \rightarrow 0$$

implies  $\text{Ker } d_{s,u}^s \in \mathcal{S}$ , and the next exact sequence

$$0 \rightarrow \text{Ker } d_{s,u}^s \rightarrow E_{s,u}^s \rightarrow \text{Im } d_{s,u}^s \rightarrow 0$$

implies  $E_{s,u}^s \in \mathcal{S}$ . By repeating this process, we obtain

$$\mathrm{Tor}_s^R(N, H_a^i(X)) \cong E_{s,u}^2 \in \mathcal{S}.$$

The proof is complete.

**Corollary 1** Let  $X$  be in  $D^+(R)$  and  $t \geq \inf X$ .

- (i) If  $\mathrm{Tor}_{-i}^R(R/\mathfrak{a}, X) \in \mathcal{S}$  and  $\mathrm{Tor}_{-i+1-i}^R(R/\mathfrak{a}, H_a^i(X)) \in \mathcal{S}$  for all  $i < t$ , then  $R/\mathfrak{a} \otimes_R H_a^t(X)$  is in  $\mathcal{S}$ .
- (ii) If  $\mathrm{Tor}_{-i+1}^R(R/\mathfrak{a}, X) \in \mathcal{S}$  and  $\mathrm{Tor}_{-i+2-i}^R(R/\mathfrak{a}, H_a^i(X)) \in \mathcal{S}$  for all  $i < t$ , then  $\mathrm{Tor}_1^R(R/\mathfrak{a}, H_a^t(X))$  is in  $\mathcal{S}$ .
- (iii) If  $\mathrm{Tor}_{-i+2}^R(R/\mathfrak{a}, X) \in \mathcal{S}$  and  $\mathrm{Tor}_{-i+3-i}^R(R/\mathfrak{a}, H_a^i(X)) \in \mathcal{S}$  for all  $i < t$ , then  $R/\mathfrak{a} \otimes_R H_a^t(X) \in \mathcal{S}$  if and only if  $\mathrm{Tor}_2^R(R/\mathfrak{a}, H_a^t(X)) \in \mathcal{S}$ .

Assume that  $\mathcal{S}$  is closed under taking direct limits. For any  $X \in D^+(R) \cap D^f(R)$ , we can define that  $t_{\mathcal{S}}(\mathfrak{a}, X) = \sup\{i \in \mathbb{Z} \mid H_a^i(X) \notin \mathcal{S}\}$  by Corollary 4.10 (Yang et al., 2025). As an application of Corollary 1, we bring the following result which is the non-finiteness on the top non-zero local cohomology modules for complexes.

**Corollary 2** Assume that  $\mathcal{S}$  is closed under taking direct limits and  $X \in D^+(R) \cap D^f(R)$ . If  $t_{\mathcal{S}}(\mathfrak{a}, X) > 0$ , then  $H_a^{t_{\mathcal{S}}(\mathfrak{a}, X)}(X)/T$  is not finite for any submodule  $T$  of  $H_a^{t_{\mathcal{S}}(\mathfrak{a}, X)}(X)$  with  $T \in \mathcal{S}$ . In particular,  $H_a^{t_{\mathcal{S}}(\mathfrak{a}, X)}(X)$  is not finite.

**Proof** Assume contrarily that  $H_a^{t_{\mathcal{S}}(\mathfrak{a}, X)}(X)/T$  is finite. Then there exists an integer  $j$  such that  $\alpha^j(H_a^{t_{\mathcal{S}}(\mathfrak{a}, X)}(X)/T) = 0$ ; that is,  $\alpha^j H_a^{t_{\mathcal{S}}(\mathfrak{a}, X)}(X) \subseteq T$ . On the other hand, by Corollary 1, we have that  $H_a^{t_{\mathcal{S}}(\mathfrak{a}, X)}(X)/\alpha^j H_a^{t_{\mathcal{S}}(\mathfrak{a}, X)}(X)$  is in  $\mathcal{S}$  and so its quotient  $H_a^{t_{\mathcal{S}}(\mathfrak{a}, X)}(X)/T$  is in  $\mathcal{S}$ . Therefore  $H_a^{t_{\mathcal{S}}(\mathfrak{a}, X)}(X)$  is in  $\mathcal{S}$  which contradicts the definition of  $t_{\mathcal{S}}(\mathfrak{a}, X)$ .

**Proposition 1** Let  $\mathfrak{a}$  be an ideal of  $R$  and  $X \in D^+(R)$  such that  $\mathrm{Tor}_j^R(R/\mathfrak{a}, H_a^i(X)) \in \mathcal{S}$  for all  $i$  and all  $j$  (resp. for  $i \leq c$  and all  $j$ ). Then  $\mathrm{Tor}_i^R(R/\mathfrak{a}, X) \in \mathcal{S}$  for all  $i$  (resp. for  $i \leq c$ ).

**Proof** We may assume that  $c \geq \inf X$ . Consider the spectral sequence (1). The hypothesis implies  $E_{p,q}^2 \in \mathcal{S}$  for all  $p$  and all  $q$  (resp. for  $\inf X \leq q \leq c$  and all  $j$ ). For all  $i$  (resp. for  $\inf X \leq i \leq c$ ), there is a finite filtration

$$0 = U^{-1} \subseteq U^0 \subseteq \dots \subseteq U^{i - \inf X} = \mathrm{Tor}_i^R(R/\mathfrak{a}, X)$$

such that  $U^p/U^{p-1} \cong E_{p,i-p}^\infty$  for  $i \geq p + \inf X$ . Since  $E_{p,i-p}^\infty$  is a subquotient of  $E_{p,i-p}^2$  for all  $i - p$  (resp. for  $\inf X \leq i - p \leq c$ ), it follows that  $E_{p,i-p}^\infty \in \mathcal{S}$  for  $i \geq p + \inf X$ . A successive use of the short exact sequence

$$0 \rightarrow U^{p-1} \rightarrow U^p \rightarrow U^p/U^{p-1} \rightarrow 0$$

implies  $\mathrm{Tor}_i^R(R/\mathfrak{a}, X) \in \mathcal{S}$ .

The following theorem reveals some sufficient conditions so that the isomorphism

$$\mathrm{Tor}_{s-t}^R(R/\mathfrak{a}, X) \cong \mathrm{Tor}_s^R(R/\mathfrak{a}, H_a^t(X))$$

holds.

**Theorem 2** Let  $X$  be in  $D^+(R)$  and  $s \geq 0, t \geq \inf X$  such that

- (i)  $\mathrm{Tor}_{s-t+i}^R(R/\mathfrak{a}, X) = 0$  for all  $\inf X \leq i < t$  or  $t + 1 \leq i \leq c + t$ .
- (ii)  $\mathrm{Tor}_{s-i+1}^R(R/\mathfrak{a}, H_a^{t-i}(X)) = 0$  for all  $1 \leq i \leq t - \inf X$ .
- (iii)  $\mathrm{Tor}_{s+i-1}^R(R/\mathfrak{a}, H_a^{t+i}(X)) = 0$  for all  $1 \leq i \leq c$ .

Then  $\mathrm{Tor}_{s-t}^R(R/\mathfrak{a}, X) \cong \mathrm{Tor}_s^R(R/\mathfrak{a}, H_a^t(X))$ .

**Proof** We may assume that  $t - \inf X \leq c$ . Set  $u = c - t$  and consider the spectral sequence (1). There is a finite filtration

$$0 = U^{-1} \subseteq U^0 \subseteq \dots \subseteq U^{s+u-\inf X} = \mathrm{Tor}_{s-t}^R(R/\mathfrak{a}, X)$$

such that  $U^p/U^{p-1} \cong E_{p,s+u-p}^\infty$  for  $s+u \geq p + \inf X$ . Let  $r \geq 2$ . Consider the differential

$$E_{s+r,u-r+1}^r \xrightarrow{d_{s+r,u-r+1}^r} E_{s,u}^r \xrightarrow{d_{s,u}^r} E_{s-r,u+r-1}^r.$$

By conditions (ii) and (iii), we have  $E_{s+r,u-r+1}^r = 0 = E_{s-r,u+r-1}^r$  for  $r \geq 2$ . As  $E_{p,s+u-p}^\infty$  is a subquotient of  $E_{p,s+u-p}^2 = \mathrm{Tor}_p^R(R/\mathfrak{a}, H_a^{s+t+p}(X))$ , we have  $U^p/U^{p-1} \cong E_{p,s+u-p}^\infty = 0$  for  $0 \leq p \leq s-1$  by conditions (i) and (iii), and  $U^s/U^{s-1} \cong E_{s,u}^\infty$  and  $U^p/U^{p-1} \cong E_{p,s+u-p}^\infty = 0$  for  $s+1 \leq p \leq s+u-\inf X$  by conditions (i) and (ii), it follows that  $0 = U^{-1} = \dots = U^{s-1}$  and  $U^s = \dots = U^{s+u-\inf X} = \mathrm{Tor}_{s-t}^R(R/\mathfrak{a}, X)$ . So

$$\text{Tor}_s^R(R/\mathfrak{a}, H_a^t(X)) = E_{s,u}^2 \cong E_{s,u}^\infty \cong U^s = \text{Tor}_{s-t}^R(R/\mathfrak{a}, X).$$

We get the isomorphism we seek.

**Corollary 3** Let  $N$  be an  $R$ -module with  $\text{Supp}_R N \subseteq V(\mathfrak{a})$ , and let  $X$  be in  $D^+(R)$  and  $s \geq 0, t \geq \inf X$  such that

- (i)  $\text{Tor}_{s-t+i}^R(N, X) = 0$  for all  $\inf X \leq i < t$  or  $t + 1 \leq i \leq c + t$ .
- (ii)  $\text{Tor}_{s-i+1}^R(N, H_a^{t-i}(X)) = 0$  for all  $1 \leq i \leq t - \inf X$ .
- (iii)  $\text{Tor}_{s+i-1}^R(N, H_a^{t+i}(X)) = 0$  for all  $1 \leq i \leq c$ .

Then  $\text{Tor}_{s-t}^R(N, X) \cong \text{Tor}_s^R(N, H_a^t(X))$ .

**Proof** Since  $\text{Supp}_R N \subseteq V(\mathfrak{a})$ , there is a finite filtration of submodules of  $N$

$$0 = N_0 \subseteq N_1 \subseteq \dots \subseteq N_r = N,$$

such that  $N_i/N_{i-1} \cong R/\mathfrak{a}$  for any  $1 \leq i \leq r$ . By Theorem 2, we have  $\text{Tor}_{s-t}^R(R/\mathfrak{a}, X) \cong \text{Tor}_s^R(R/\mathfrak{a}, H_a^t(X))$ . Since there is the short exact sequence

$$0 \rightarrow N_0 \rightarrow N_1 \rightarrow N_1/N_0 \rightarrow 0,$$

such that  $\text{Tor}_{s-t}^R(N_1, X) \cong \text{Tor}_s^R(N_1, H_a^t(X))$ . Again, by the short exact sequence

$$0 \rightarrow N_1 \rightarrow N_2 \rightarrow N_2/N_1 \rightarrow 0,$$

one has  $\text{Tor}_{s-t}^R(N_2, X) \cong \text{Tor}_s^R(N_2, H_a^t(X))$ . In the following, we reuse the short exact sequence

$$0 \rightarrow N_{i-1} \rightarrow N_i \rightarrow N_i/N_{i-1} \rightarrow 0,$$

then  $\text{Tor}_{s-t}^R(N, X) \cong \text{Tor}_s^R(N, H_a^t(X))$ .

**Corollary 4** Let  $Y$  be a bounded  $R$ -complex with  $\text{Supp}_R Y \subseteq V(\mathfrak{a})$ , and let  $X$  be in  $D^+(R)$  and  $s \geq \sup Y, t \geq \inf X$  such that

- (i)  $\text{Tor}_{s-t+i}^R(Y, X) = 0$  for all  $\inf X \leq i < t$  or  $t + 1 \leq i \leq c + t - \sup Y$ .
- (ii)  $\text{Tor}_{s-i+1}^R(Y, H_a^{t-i}(X)) = 0$  for all  $1 \leq i \leq t - \inf X$ .
- (iii)  $\text{Tor}_{s+i-1}^R(Y, H_a^{t+i}(X)) = 0$  for all  $1 \leq i \leq c - \sup Y$ .

Then  $\text{Tor}_{s-t-\sup Y}^R(Y, X) \cong \text{Tor}_{s-\sup Y}^R(Y, H_a^t(X))$ .

**Proof** Induction on  $\sup Y - \inf Y$ . If  $\inf Y = \sup Y = r$ , then  $Y \simeq \Sigma^r H_r(Y)$ . Since Corollary 3, one has  $\text{Tor}_{s-t-\sup Y}^R(Y, X) \cong \text{Tor}_{s-t}^R(H_r(Y), X) \cong \text{Tor}_{s-\sup Y}^R(Y, H_a^t(X))$ . Now assume  $\sup Y - \inf Y > 0$ . There is a distinguished triangle

$$\Sigma^{\sup Y} H_{\sup Y}(Y) \rightarrow Y \rightarrow Y' \rightarrow \Sigma^{\sup Y+1} H_{\sup Y}(Y)$$

in  $D(R)$ , where  $Y' = 0 \rightarrow Y_{\sup Y}/\text{Ker } d_{\sup Y} \xrightarrow{\bar{d}_{\sup Y}} Y_{\sup Y-1} \xrightarrow{d_{\sup Y-1}} \dots$ . This induces exact sequence

$$\text{Tor}_{s-t-\sup Y}^R(\Sigma^{\sup Y} H_{\sup Y}(Y), X) \rightarrow \text{Tor}_{s-t-\sup Y}^R(Y, X) \rightarrow \text{Tor}_{s-t-\sup Y}^R(Y', X),$$

so that, by the inductive hypothesis, we have  $\text{Tor}_{s-t-\sup Y}^R(Y, X) \cong \text{Tor}_{s-\sup Y}^R(Y, H_a^t(X))$ .

For an  $R$ -complex  $X$  and  $m \in \mathbb{Z}$ , the  $m$ -th Betti number  $\beta_m^R(\mathfrak{p}, X)$  of  $X$  is defined to be the dimension of the  $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ -vector space  $\text{Tor}_m^R(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, X_{\mathfrak{p}})$ .

**Corollary 5** Let  $\mathfrak{p} \in V(\mathfrak{a}), X \in D^+(R)$  and  $s \geq 0, t \geq \inf X$  such that

- (i)  $\text{Tor}_{s-t+i}^R(R/\mathfrak{p}, X) = 0$  for all  $\inf X \leq i < t$  or  $t + 1 \leq i \leq c + t$ .
- (ii)  $\text{Tor}_{s-i+1}^R(R/\mathfrak{p}, H_a^{t-i}(X)) = 0$  for all  $1 \leq i \leq t - \inf X$ .
- (iii)  $\text{Tor}_{s+i-1}^R(R/\mathfrak{p}, H_a^{t+i}(X)) = 0$  for all  $1 \leq i \leq c$ .

Then  $\beta_{s-t}^R(\mathfrak{p}, X) = \beta_s^R(\mathfrak{p}, H_a^t(X))$ .

**Proof** Since  $\mathfrak{p} \in V(\mathfrak{a})$  and Theorem 2, we have  $\text{Tor}_{s-t}^R(R/\mathfrak{p}, X) \cong \text{Tor}_s^R(R/\mathfrak{p}, H_a^t(X))$ . Hence  $\text{Tor}_{s-t}^R(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, X_{\mathfrak{p}}) \cong \text{Tor}_s^R(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, H_a^t(X)_{\mathfrak{p}})$  by Lemma 6. 1. 4 (Christensen et al., 2006). According to the definition of Betti number of  $R$ -complex  $X$ , we have  $\beta_{s-t}^R(\mathfrak{p}, X) = \beta_s^R(\mathfrak{p}, H_a^t(X))$ .

For an  $R$ -complex  $X$ , the flat dimension  $\text{fd}_R X$  is defined as

$\text{fd}_R X = \inf \{ n \in \mathbb{Z} \mid \exists \text{ a semiflat } R\text{-complex } F \text{ such that } F \simeq X \text{ and } F_v = 0 \text{ for all } v > n \}.$

**Corollary 6** An equality

$$\text{fd}_R X = \sup \left\{ m \in \mathbb{Z} \mid \exists \mathfrak{p} \in \text{Spec } R : \beta_m^R(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, H_a^i(X)_{\mathfrak{p}}) \neq 0 \right\} + t.$$

**Proof** Follows from the Lemma 6. 1. 15 (Christensen et al., 2006) and Corollary 5.

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## 复形的局部上调模的 torsion 函子

张平儿<sup>1</sup>, 马亚军<sup>2</sup>

1. 甘肃政法大学人工智能学院, 甘肃 兰州 730070

2. 兰州交通大学数理学院, 甘肃 兰州 730070

**摘要:** 证明了下有界复形的局部上调模的 torsion 模属于 Serre 类, 并且在某种情形下证明了复形的 torsion 模与局部上调模的 torsion 模的同构式. 作为此同构式的应用, 我们用复形的局部上调模表示了复形的 Betti 数.

**关键词:** 局部上调模; torsion 函子; Serre 子范畴; Betti 数

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