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三维稳态热带气候模型各向异性的Liouville定理*

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摘　要： 为了使三维稳态热带气候模型只有平凡的零解，利用能量估计方法以及Sobolev嵌入得到了使其成立的充分条件，其中关于速度场的可积性条件涉及到各向异性．所得结论推广了已有的关于三维稳态热带气候模型Liouville定理的结果．

关键词： 稳态热带气候模型；Liouville定理；各向异性

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Anisotropic Liouville theorem for the three-dimensional stationary tropical climate model

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Abstract: In order to make the three-dimensional steady-state tropical climate model only have the trivial zero solution, sufficient conditions are obtained by using the energy estimation and Sobolev embedding, in which the integrability of the velocity field involves the anisotropy. The result obtained extends the known results with respect to Liouville theorem for three-dimensional steady-state tropical climate model.

Key words: stationary tropical climate model; Liouville theorem; anisotropy

本文研究三维稳态热带气候模型的Liouville定理，其模型如下

$$\begin{cases} u\cdot\nabla u+\operatorname{div}(v\otimes v)-\Delta u+\nabla P=0,\\ u\cdot\nabla v+v\cdot\nabla u-\Delta v+\nabla\theta=0,\\ u\cdot\nabla\theta-\Delta\theta+\nabla\cdot v=0,\\ \nabla\cdot u=0,\end{cases}\tag{1}$$

其中$u=u(x)=(u_1(x),u_2(x),u_3(x))$和$v=v(x)=(v_1(x),v_2(x),v_3(x))$分别表示速度场的正压模式和第一斜压模式．标量函数$P=P(x)$和$\theta=\theta(x)$分别表示压力和温度．矩阵$v\otimes v$的分量为$v_iv_j, i,j=1,2,3$.

热带气候模型作为速度正压模式和第一斜压模式与典型中层温度之间的耦合系统，它比Navier-Stokes系统具有更丰富的结构．当$v=0$，$\theta=0$时，系统(1)退化为如下三维稳态不可压缩Navier-Stokes系统

$$\begin{cases} u\cdot\nabla u-\Delta u+\nabla P=0,\\ \nabla\cdot u=0.\end{cases}\tag{2}$$

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关于系统(2)的Liouville定理，Galdi(2014)证明了当 $u \in L^{\frac{9}{2}}(\mathbb{R}^3)$ 时，$u \equiv 0$. Chae et al.(2016)对Galdi(2014)的结果进行了对数改进，即证明了如果

$$\int_{\mathbb{R}^3} |u(x)|^{\frac{9}{2}} \left\{ \log\left(2 + \frac{1}{|u(x)|}\right) \right\}^{-1} dx < +\infty,$$

那么 $u = 0$. Chae(2014)得出若 $\Delta u \in L^{\frac{6}{5}}(\mathbb{R}^3)$，则系统(2)具有唯一的平凡解. Seregin(2016)证明了若 $u \in L^6(\mathbb{R}^3) \cap \mathrm{BMO}^{-1}(\mathbb{R}^3)$，系统(2)的Liouville定理仍然成立. 最近，Chae(2023)给出了如下充分条件

$$u \in L^6(\mathbb{R}^3) \cap L^q(\mathbb{R}^3), \quad u_i \in L_{x_i}^{\frac{q}{q-2}} L_{\tilde{x}_i}^s (\mathbb{R} \times \mathbb{R}^2)\ (i = 1, 2, 3), \frac{2}{q} + \frac{1}{s} \geqslant \frac{1}{2}, 1 \leqslant s \leqslant +\infty, 2 < q < +\infty, \tag{3}$$

其中 $L_{x_i}^{\frac{q}{q-2}} L_{\tilde{x}_i}^s (\mathbb{R} \times \mathbb{R}^2)$ 表示各向异性空间，其范数见定义1.

目前，三维稳态热带气候模型引起了越来越多的关注(原保全等,2022;祖倩等,2023;Li et al., 2019; Chaharlang et al.,2020). 对于该模型的Liouville定理，Ding et al.(2021)得出如下充分条件

$$u \in L^q(\mathbb{R}^3), (v, \theta) \in L^3(\mathbb{R}^3) \cap L^q(\mathbb{R}^3), 3 \leqslant q \leqslant \frac{9}{2}. \tag{4}$$

最近，在局部Morrey空间中，Yuan et al.(2023)证明了若

$$(u, v, \theta) \in M_{\gamma,0}^p(\mathbb{R}^3),\ v \in M_{\gamma,0}^2(\mathbb{R}^3),\ \theta \in M_\gamma^2(\mathbb{R}^3),\ 3 \leqslant p < \frac{9}{2},\ 0 < \gamma \leqslant 1,\ 3\gamma + 2p \leqslant 9,$$

则有 $u = v = 0$，$\theta = 0$；Ding et al.(2023)给出了如下充分条件

$$\begin{cases} (u, v, \theta) \in M_{\gamma,0}^p(\mathbb{R}^3), (v, \theta) \in L^r(\mathbb{R}^3), 0 < \gamma < 3 \leqslant p < +\infty, \\ \dfrac{\gamma}{3} \sum_{i=1}^3 \dfrac{1}{p_i} - \sum_{i=1}^3 \dfrac{1}{p_i} + \dfrac{2}{3} \leqslant 0. \end{cases}$$

另外，关于更多模型的Liouville定理，可参见文献(王科研等,2021;田琴等,2023;周艳平等,2023).

受以上文献启发，本文研究在速度分量各向异性可积性条件下的三维稳态热带气候模型的Liouville定理，并得到如下结果.

定理1　设 (u, v, θ) 是系统(1)的光滑解，若

$$u \in L^6(\mathbb{R}^3) \cap L^q(\mathbb{R}^3),\ v \in L^{\frac{3m}{2m-3}}(\mathbb{R}^3) \cap L^q(\mathbb{R}^3),\ \theta \in L^m(\mathbb{R}^3),\ u_i \in L_{x_i}^{\frac{q}{q-2}} L_{\tilde{x}_i}^s(\mathbb{R} \times \mathbb{R}^2)\quad (i = 1, 2, 3),$$

其中 q, s, m 满足

$$\frac{2}{q} + \frac{1}{s} \geqslant \frac{1}{2},\quad 1 \leqslant s \leqslant +\infty,\quad 2 < q < +\infty,\quad \frac{12}{5} \leqslant m \leqslant 4, \tag{5}$$

则 $u = v = 0$，$\theta = 0$.

注1　当 $v = 0$，$\theta = 0$ 时，系统(1)退化为经典的Navier-Stokes系统，此时定理1中的充分条件正好是条件(3). 因此，定理1将文献Chae(2023)中关于Navier-Stokes系统的Liouville定理推广到了热带气候模型.

注2　当 $q = s = 3$ 时，定理1中关于 (u, v, θ) 的条件为

$$u \in L^6(\mathbb{R}^3) \cap L^3(\mathbb{R}^3),\quad v \in L^{\frac{3m}{2m-3}}(\mathbb{R}^3) \cap L^3(\mathbb{R}^3),\quad \theta \in L^m(\mathbb{R}^3),\quad \frac{12}{5} \leqslant m \leqslant 4. \tag{6}$$

相比较于条件(4)，二者无包含关系.

1　预备知识

下面介绍本文将要使用的定义和引理. 首先，记

$$\tilde{x}_1 := (x_2, x_3),\ \tilde{x}_2 := (x_1, x_3),\ \tilde{x}_3 := (x_1, x_2).$$

对于 $\Omega \subset \mathbb{R}^2$，定义

$$\int_\Omega f\, d\tilde{x}_1 := \int_\Omega f\, dx_2 dx_3,\quad \int_\Omega f\, d\tilde{x}_2 := \int_\Omega f\, dx_1 dx_3,\quad \int_\Omega f\, d\tilde{x}_3 := \int_\Omega f\, dx_1 dx_2.$$

定义 1 当$1 \leqslant r, s \leqslant +\infty$时，若

$$\|f\|_{L_{x_i}^r L_{\tilde{x}_i}^s(\mathbb{R}\times\mathbb{R}^2)} := \left\{\int_{\mathbb{R}}\left(\int_{\mathbb{R}^2}|f|^s \mathrm{d}\tilde{x}_i\right)^{\frac{r}{s}} \mathrm{d}x_i\right\}^{\frac{1}{r}} < +\infty \quad (i = 1, 2, 3),$$

称函数$f \in L_{x_i}^r L_{\tilde{x}_i}^s(\mathbb{R}\times\mathbb{R}^2)$. 当$r = +\infty$或$s = +\infty$时，理解为本性上确界.

下面引入截断函数的定义. 令

$$\mathbb{J}_1 = \{2, 3\}, \ \mathbb{J}_2 = \{1, 3\}, \ \mathbb{J}_3 = \{1, 2\}.$$

当$R > 0$时，记

$$D_i := \left\{\tilde{x}_i \in \mathbb{R}^2 \,\Big|\, |x_j| < 2R, \ j \in \mathbb{J}_i\right\}, \ i \in \{1, 2, 3\}.$$

定义 2 设光滑非增函数$\psi: [0, +\infty) \to [0, 1]$，满足

$$\psi(s) = \begin{cases} 1, & 0 \leqslant s \leqslant 1, \\ 0, & s \geqslant 4. \end{cases}$$

定义截断函数$\phi_R(x)$如下

$$\phi_R(x) = \prod_{j=1}^{3} \psi\left(\frac{x_j^2}{R^2}\right).$$

令

$$\tilde{\phi}_{i,R}(x) = \prod_{j \in \mathbb{J}_i} \psi\left(\frac{x_j^2}{R^2}\right), \ i \in \{1, 2, 3\},$$

则该函数满足如下性质$\|\nabla^k \phi_R\|_{L^\infty} \leqslant CR^{-k}$，$k = 0, 1, 2$，$C$为常数，

$$\Delta\phi_R = \sum_{i=1}^{3} \partial_{x_i x_i} \phi_R = \sum_{i=1}^{3} \left[\frac{2}{R^2}\psi'\left(\frac{x_i^2}{R^2}\right) + \frac{4x_i^2}{R^4}\psi''\left(\frac{x_i^2}{R^2}\right)\right]\tilde{\phi}_{i,R}.$$

引理 1（丁勇，2013） 设$1 < q < +\infty$，$R_i = \dfrac{\partial_i}{\sqrt{-\Delta}}(1 \leqslant i \leqslant n)$是$\mathbb{R}^n$上的Riesz变换，存在常数$C > 0$，使得对任意$f \in L^q(\mathbb{R}^n)$，有

$$\|R_i f\|_{L^q(\mathbb{R}^n)} \leqslant C\|f\|_{L^q(\mathbb{R}^n)}.$$

2 主要结果证明

证明 将系统(1)的第1个式子两边同时乘以$u\phi_R$，并在$\mathbb{R}^3$上积分，有

$$\int_{\mathbb{R}^3} (u\cdot\nabla u)\cdot u\phi_R + \nabla P\cdot u\phi_R + \mathrm{div}(v\otimes v)\cdot u\phi_R - \Delta u\cdot u\phi_R \,\mathrm{d}x = 0. \tag{7}$$

对式(7)每一项利用分部积分，并由$\nabla\cdot u = 0$，得

$$\int_{\mathbb{R}^3} (u\cdot\nabla u)\cdot u\phi_R \,\mathrm{d}x = \int_{\mathbb{R}^3} u_j\partial_j u_i \phi_R u_i \,\mathrm{d}x = \int_{\mathbb{R}^3} u_j\phi_R \partial_j\left(\frac{u_i^2}{2}\right)\mathrm{d}x = -\int_{\mathbb{R}^3} u_j\partial_j\phi_R\left(\frac{u_i^2}{2}\right)\mathrm{d}x = -\int_{\mathbb{R}^3}\nabla\phi_R\cdot u\frac{|u|^2}{2}\mathrm{d}x. \tag{8}$$

$$\begin{aligned}\int_{\mathbb{R}^3} \Delta u\cdot\phi_R u\,\mathrm{d}x &= \int_{\mathbb{R}^3} \partial_j^2 u_i\phi_R u_i\,\mathrm{d}x = -\int_{\mathbb{R}^3}\partial_j u_i\partial_j\phi_R u_i\,\mathrm{d}x - \int_{\mathbb{R}^3}\partial_j u_i\phi_R\partial_j u_i\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^3}\partial_j\left(\frac{u_i^2}{2}\right)\partial_j\phi_R\,\mathrm{d}x - \int_{\mathbb{R}^3}(\partial_j u_i)^2\phi_R\,\mathrm{d}x = \int_{\mathbb{R}^3}\left(\frac{u_i^2}{2}\right)\partial_j^2\phi_R\,\mathrm{d}x - \int_{\mathbb{R}^3}(\partial_j u_i)^2\phi_R\,\mathrm{d}x \\ &= \int_{\mathbb{R}^3}\Delta\phi_R\cdot\frac{|u|^2}{2}\mathrm{d}x - \int_{\mathbb{R}^3}\phi_R\cdot|\nabla u|^2\,\mathrm{d}x.\end{aligned} \tag{9}$$

$$\int_{\mathbb{R}^3}\nabla P\cdot\phi_R u\,\mathrm{d}x = \int_{\mathbb{R}^3}\partial_i P\phi_R u_i\,\mathrm{d}x = -\int_{\mathbb{R}^3}P\partial_i\phi_R u_i\,\mathrm{d}x = -\int_{\mathbb{R}^3}\nabla\phi_R\cdot(Pu)\,\mathrm{d}x. \tag{10}$$

$$\begin{aligned}\int_{\mathbb{R}^3}\mathrm{div}(v\otimes v)\cdot\phi_R u\,\mathrm{d}x &= \int_{\mathbb{R}^3}\partial_j(v_i v_j)\phi_R u_i\,\mathrm{d}x = -\int_{\mathbb{R}^3}v_i v_j\partial_j\phi_R u_i\,\mathrm{d}x - \int_{\mathbb{R}^3}v_i v_j\phi_R\partial_j u_i\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^3}\nabla\phi_R(v\otimes v)\cdot u\,\mathrm{d}x - \int_{\mathbb{R}^3}\phi_R(v\otimes v)\cdot\nabla u\,\mathrm{d}x.\end{aligned} \tag{11}$$

将式(8)~(11)代入式(7)中，有

$$\int_{\mathbb{R}^3}\phi_R|\nabla u|^2\mathrm{d}x=\int_{\mathbb{R}^3}\Delta\phi_R\frac{|u|^2}{2}\mathrm{d}x+\int_{\mathbb{R}^3}\nabla\phi_R\cdot\frac{|u|^2}{2}u\,\mathrm{d}x+\int_{\mathbb{R}^3}\nabla\phi_R\cdot Pu\,\mathrm{d}x$$
$$+\int_{\mathbb{R}^3}\phi_R\,(v\otimes v)\cdot\nabla u\,\mathrm{d}x+\int_{\mathbb{R}^3}\nabla\phi_R(v\otimes v)\cdot u\,\mathrm{d}x. \tag{12}$$

将系统(1)的第2个和第3个式子两边分别乘以$v\phi_R$和$\theta\phi_R$，并在$\mathbb{R}^3$上积分，得

$$\int_{\mathbb{R}^3}\phi_R|\nabla v|^2\mathrm{d}x=\int_{\mathbb{R}^3}\Delta\phi_R\frac{|v|^2}{2}\mathrm{d}x+\int_{\mathbb{R}^3}\nabla\phi_R\cdot\frac{|v|^2}{2}u\,\mathrm{d}x+\int_{\mathbb{R}^3}\nabla\phi_R\cdot\theta\,v\,\mathrm{d}x$$
$$+\int_{\mathbb{R}^3}\mathrm{div}v\cdot\phi_R\theta\,\mathrm{d}x-\int_{\mathbb{R}^3}\phi_R\,(v\otimes v)\cdot\nabla u\,\mathrm{d}x. \tag{13}$$

$$\int_{\mathbb{R}^3}\phi_R|\nabla\theta|^2\mathrm{d}x=\int_{\mathbb{R}^3}\Delta\phi_R\frac{|\theta|^2}{2}\mathrm{d}x+\int_{\mathbb{R}^3}\nabla\phi_R\cdot\frac{|\theta|^2}{2}u\,\mathrm{d}x-\int_{\mathbb{R}^3}\mathrm{div}v\cdot\phi_R\theta\,\mathrm{d}x. \tag{14}$$

将式(12)~(14)相加，得

$$\int_{\mathbb{R}^3}\left(|\nabla u|^2+|\nabla v|^2+|\nabla\theta|^2\right)\phi_R\,\mathrm{d}x=\frac{1}{2}\int_{\mathbb{R}^3}|u|^2\Delta\phi_R\,\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^3}|v|^2\Delta\phi_R\,\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^3}|\theta|^2\Delta\phi_R\,\mathrm{d}x$$
$$+\int_{\mathbb{R}^3}\left(\frac{1}{2}|u|^2+\frac{1}{2}|v|^2+P\right)u\cdot\nabla\phi_R\,\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^3}|\theta|^2u\cdot\nabla\phi_R\,\mathrm{d}x$$
$$+\int_{\mathbb{R}^3}\nabla\phi_R(v\otimes v)\cdot u\,\mathrm{d}x+\int_{\mathbb{R}^3}v\,\theta\cdot\nabla\phi_R\,\mathrm{d}x=:\sum_{k=1}^{7}I_k. \tag{15}$$

下面分别估计I_1-I_7. 对I_1，由Hölder不等式，有

$$I_1=\frac{1}{2}\sum_{i=1}^{3}\int_{R\leqslant|x_i|\leqslant 2R}\int_{D_i}|u|^2\tilde{\phi}_{i,R}\left(\frac{2}{R^2}\psi'\left(\frac{x_i^2}{R^2}\right)+\frac{4x_i^2}{R^4}\psi''\left(\frac{x_i^2}{R^2}\right)\right)\mathrm{d}\tilde{x}_i\mathrm{d}x_i$$
$$\leqslant\frac{C}{R^2}\sum_{i=1}^{3}\left(\int_{R\leqslant|x_i|\leqslant 2R}\int_{D_i}\left(|u|^2\right)^3\mathrm{d}x\right)^{\frac{1}{3}}\left(\int_{R\leqslant|x_i|\leqslant 2R}\int_{D_i}1\,\mathrm{d}x\right)^{\frac{2}{3}}\leqslant C\sum_{i=1}^{3}\left(\int_{R\leqslant|x_i|\leqslant 2R}\int_{\mathbb{R}^2}|u|^6\,\mathrm{d}x\right)^{\frac{1}{3}}.$$

当$R\to+\infty$时，$I_1\to 0$.

类似地，对I_2，利用Hölder不等式，有

$$I_2=\frac{1}{2}\sum_{i=1}^{3}\int_{R\leqslant|x_i|\leqslant 2R}\int_{D_i}|v|^2\tilde{\phi}_{i,R}\left(\frac{2}{R^2}\psi'\left(\frac{x_i^2}{R^2}\right)+\frac{4x_i^2}{R^4}\psi''\left(\frac{x_i^2}{R^2}\right)\right)\mathrm{d}\tilde{x}_i\mathrm{d}x_i$$
$$\leqslant\frac{C}{R^2}\sum_{i=1}^{3}\left(\int_{R\leqslant|x_i|\leqslant 2R}\int_{D_i}\left(|v|^2\right)^{\frac{3m}{2(2m-3)}}\mathrm{d}x\right)^{\frac{2(2m-3)}{3m}}\left(\int_{R\leqslant|x_i|\leqslant 2R}\int_{D_i}1\,\mathrm{d}x\right)^{1-\frac{2(2m-3)}{3m}}$$
$$\leqslant CR^{\frac{6}{m}-3}\sum_{i=1}^{3}\left(\int_{R\leqslant|x_i|\leqslant 2R}\int_{\mathbb{R}^2}|v|^{\frac{3m}{(2m-3)}}\mathrm{d}x\right)^{\frac{2(2m-3)}{3m}}.$$

由条件(5)知$\frac{6}{m}-3<0$. 因此，当$R\to+\infty$时，$I_2\to 0$. I_3与I_2的估计方法类似，

$$I_3=\frac{1}{2}\sum_{i=1}^{3}\int_{R\leqslant|x_i|\leqslant 2R}\int_{D_i}|\theta|^2\tilde{\phi}_{i,R}\left(\frac{2}{R^2}\psi'\left(\frac{x_i^2}{R^2}\right)+\frac{4x_i^2}{R^4}\psi''\left(\frac{x_i^2}{R^2}\right)\right)\mathrm{d}\tilde{x}_i\mathrm{d}x_i$$
$$\leqslant\frac{C}{R^2}\sum_{i=1}^{3}\left(\int_{R\leqslant|x_i|\leqslant 2R}\int_{D_i}\left(|\theta|^2\right)^{\frac{m}{2}}\mathrm{d}x\right)^{\frac{2}{m}}\left(\int_{R\leqslant|x_i|\leqslant 2R}\int_{D_i}1\,\mathrm{d}x\right)^{1-\frac{2}{m}}\leqslant CR^{3\left(1-\frac{2}{m}\right)-2}\sum_{i=1}^{3}\left(\int_{R\leqslant|x_i|\leqslant 2R}\int_{\mathbb{R}^2}|\theta|^m\,\mathrm{d}x\right)^{\frac{2}{m}}$$
$$\leqslant CR^{1-\frac{6}{m}}\sum_{i=1}^{3}\left(\int_{R\leqslant|x_i|\leqslant 2R}\int_{\mathbb{R}^2}|\theta|^m\,\mathrm{d}x\right)^{\frac{2}{m}}.$$

由条件(5)知$1-\frac{6}{m}<0$. 因此，当$R\to+\infty$时，$I_3\to 0$.

下面对I_4进行估计. 首先，将散度作用于系统(1)的第1个式子，由$R_i=\frac{\partial_i}{\sqrt{-\Delta}}$以及条件$\nabla\cdot u=0$，得

$$P=\frac{1}{-\Delta}\partial_i\partial_j\left(u_iu_j\right)+\frac{1}{-\Delta}\partial_i\partial_j\left(v_iv_j\right)=-\left(R_iR_j\left(u_iu_j\right)+R_iR_j\left(v_iv_j\right)\right).$$

当$q\in(1,+\infty)$时，根据Hölder不等式、Calderon-Zygmund 不等式以及引理1可知

$$\|P\|_{L^{\frac{q}{2}}}\leqslant C\left(\|u\|_{L^q}^2+\|v\|_{L^q}^2\right),$$

进而有

$$\left\|\frac{1}{2}|u|^2+\frac{1}{2}|v|^2+P\right\|_{L^{\frac{q}{2}}}\leqslant C\left(\|u\|_{L^q}^2+\|v\|_{L^q}^2\right).$$

因此，在速度分量各向异性的可积性条件下，有

$$\begin{aligned}
I_4&\leqslant\frac{C}{R}\sum_{i=1}^3\int_{R\leqslant|x_i|\leqslant 2R}\int_{D_i}\left|\frac{1}{2}|u|^2+\frac{1}{2}|v|^2+P\right||u_i|\,\mathrm{d}\tilde{x}_i\mathrm{d}x_i\\
&\leqslant\frac{C}{R}\sum_{i=1}^3\int_{R\leqslant|x_i|\leqslant 2R}\left(\int_{D_i}\left|\frac{1}{2}|u|^2+\frac{1}{2}|v|^2+P\right|^{\frac{q}{2}}\mathrm{d}\tilde{x}_i\right)^{\frac{2}{q}}\left(\int_{D_i}|u_i|^s\,\mathrm{d}\tilde{x}_i\right)^{\frac{1}{s}}\left(\int_{D_i}1\,\mathrm{d}\tilde{x}_i\right)^{\frac{(qs-q-2s)}{qs}}\mathrm{d}x_i\\
&\leqslant CR^{\frac{2(qs-q-2s)}{qs}-1}\sum_{i=1}^3\int_{R\leqslant|x_i|\leqslant 2R}\left(\int_{D_i}\left|\frac{1}{2}|u|^2+\frac{1}{2}|v|^2+P\right|^{\frac{q}{2}}\mathrm{d}\tilde{x}_i\right)^{\frac{2}{q}}\left(\int_{D_i}|u_i|^s\,\mathrm{d}\tilde{x}_i\right)^{\frac{1}{s}}\mathrm{d}x_i\\
&\leqslant CR^{\frac{2(qs-q-2s)}{qs}-1}\sum_{i=1}^3\left(\int_{R\leqslant|x_i|\leqslant 2R}\int_{D_i}\left|\frac{1}{2}|u|^2+\frac{1}{2}|v|^2+P\right|^{\frac{q}{2}}\mathrm{d}\tilde{x}_i\mathrm{d}x_i\right)^{\frac{2}{q}}\left(\int_{R\leqslant|x_i|\leqslant 2R}\left(\int_{D_i}|u_i|^s\,\mathrm{d}\tilde{x}_i\right)^{\frac{q}{s(q-2)}}\mathrm{d}x_i\right)^{\frac{q-2}{q}}\\
&\leqslant CR^{\frac{2(qs-q-2s)}{qs}-1}\left(\int_{\mathbb{R}^3}\left|\frac{1}{2}|u|^2+\frac{1}{2}|v|^2+P\right|^{\frac{q}{2}}\mathrm{d}x\right)^{\frac{2}{q}}\sum_{i=1}^3\left(\int_{R\leqslant|x_i|\leqslant 2R}\left(\int_{D_i}|u_i|^s\,\mathrm{d}\tilde{x}_i\right)^{\frac{q}{s(q-2)}}\mathrm{d}x_i\right)^{\frac{q-2}{q}}\\
&\leqslant CR^{\frac{2(qs-q-2s)}{qs}-1}\left(\int_{\mathbb{R}^3}\left(|u|^q+|v|^q\right)\mathrm{d}x\right)^{\frac{2}{q}}\sum_{i=1}^3\left(\int_{R\leqslant|x_i|\leqslant 2R}\left(\int_{D_i}|u_i|^s\,\mathrm{d}\tilde{x}_i\right)^{\frac{q}{s(q-2)}}\mathrm{d}x_i\right)^{\frac{q-2}{q}}.
\end{aligned}$$

由条件(5)中的$\frac{2}{q}+\frac{1}{s}\geqslant\frac{1}{2}$，得$\frac{2(qs-q-2s)}{qs}-1\leqslant 0$. 当$R\to+\infty$时，$I_4\to 0$.

对I_5，利用Hölder不等式，有

$$\begin{aligned}
I_5&\leqslant\frac{C}{R}\sum_{i=1}^3\int_{R\leqslant|x_i|\leqslant 2R}\int_{\mathbb{R}^2}|u||\theta|^2\,\mathrm{d}x\leqslant\frac{C}{R}\sum_{i=1}^3\left(\int_{R\leqslant|x_i|\leqslant 2R}\int_{D_i}(\theta^2)^{\frac{m}{2}}\mathrm{d}x\right)^{\frac{2}{m}}\left(\int_{R\leqslant|x_i|\leqslant 2R}\int_{D_i}|u|^6\,\mathrm{d}x\right)^{\frac{1}{6}}\left(\int_{R\leqslant|x_i|\leqslant 2R}\int_{D_i}1\,\mathrm{d}x\right)^{\frac{5}{6}-\frac{2}{m}}\\
&\leqslant CR^{3\left(\frac{5}{6}-\frac{2}{m}\right)-1}\sum_{i=1}^3\left(\int_{R\leqslant|x_i|\leqslant 2R}\int_{D_i}(\theta^2)^{\frac{m}{2}}\mathrm{d}x\right)^{\frac{2}{m}}\left(\int_{R\leqslant|x_i|\leqslant 2R}\int_{D_i}|u|^6\,\mathrm{d}x\right)^{\frac{1}{6}}\\
&\leqslant CR^{\frac{3}{2}-\frac{6}{m}}\sum_{i=1}^3\left(\int_{R\leqslant|x_i|\leqslant 2R}\int_{\mathbb{R}^2}(\theta^2)^{\frac{m}{2}}\mathrm{d}x\right)^{\frac{2}{m}}\left(\int_{R\leqslant|x_i|\leqslant 2R}\int_{\mathbb{R}^2}|u|^6\,\mathrm{d}x\right)^{\frac{1}{6}}.
\end{aligned}$$

在该项估计第2行中，由条件$m\geqslant\frac{12}{5}$，得$\frac{5}{6}-\frac{2}{m}\geqslant 0$. 再由条件(5)知$\frac{3}{2}-\frac{6}{m}\leqslant 0$. 因此，当$R\to+\infty$时，$I_5\to 0$.

I_6与I_4的估计方法类似，

$$I_6\leqslant CR^{\frac{2(qs-q-2s)}{qs}-1}\left(\int_{\mathbb{R}^3}(v^q)\,\mathrm{d}x\right)^{\frac{2}{q}}\sum_{i=1}^3\left(\int_{R\leqslant|x_i|\leqslant 2R}\left(\int_{D_i}|u_i|^s\,\mathrm{d}\tilde{x}_i\right)^{\frac{q}{s(q-2)}}\mathrm{d}x_i\right)^{\frac{q-2}{q}}.$$

当$R\to+\infty$时，$I_6\to 0$.

最后对I_7，有

$$
\begin{aligned}
I_7 &\leqslant \frac{C}{R}\sum_{i=1}^{3}\int_{R\leqslant|x_i|\leqslant 2R}\int_{D_i}|\theta||v|\mathrm{d}x \leqslant \frac{C}{R}\sum_{i=1}^{3}\left(\int_{R\leqslant|x_i|\leqslant 2R}\int_{D_i}|\theta|^m\,\mathrm{d}x\right)^{\frac{1}{m}}\left(\int_{R\leqslant|x_i|\leqslant 2R}\int_{D_i}|v|^{\frac{3m}{2m-3}}\mathrm{d}x\right)^{\frac{2m-3}{3m}}\left(\int_{R\leqslant|x_i|\leqslant 2R}\int_{D_i}1\,\mathrm{d}x\right)^{1-\frac{1}{m}-\frac{2m-3}{3m}} \\
&\leqslant CR^{3\left(1-\frac{1}{m}-\frac{2m-3}{3m}\right)-1}\sum_{i=1}^{3}\left(\int_{R\leqslant|x_i|\leqslant 2R}\int_{D_i}|\theta|^m\,\mathrm{d}x\right)^{\frac{1}{m}}\left(\int_{R\leqslant|x_i|\leqslant 2R}\int_{D_i}|v|^{\frac{3m}{2m-3}}\mathrm{d}x\right)^{\frac{2m-3}{3m}} \\
&\leqslant C\sum_{i=1}^{3}\left(\int_{R\leqslant|x_i|\leqslant 2R}\int_{\mathbb{R}^2}|\theta|^m\,\mathrm{d}x\right)^{\frac{1}{m}}\left(\int_{R\leqslant|x_i|\leqslant 2R}\int_{\mathbb{R}^2}|v|^{\frac{3m}{2m-3}}\mathrm{d}x\right)^{\frac{2m-3}{3m}}.
\end{aligned}
$$

当$R\to+\infty$时，$I_7\to 0$.

综上，当$R\to+\infty$时，$\sum_{k=1}^{7}I_k\to 0$，即$\int_{\mathbb{R}^3}\left(|\nabla u|^2+|\nabla v|^2+|\nabla\theta|^2\right)\mathrm{d}x=0$，从而有$u=v=0$，$\theta=0$. 定理1得证.

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