DOI:10. 13471/j. cnki. acta. snus. ZR20240034

A congruence related to central trinomial coefficients*

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Abstract: The central trinomial coefficient T_n denotes the coefficient of x^n in the expansion of $(1+x+x^2)^n$. We prove a congruence related to the sums of the central trinomial coefficient and the central binomial coefficient, which was conjectured by Z.-W. Sun.

Key words: supercongruences; central trinomial coefficients; central binomial coefficients; Fermat quotient; Legendre symbol

CLC number: O156.1 **Document code:** A **Article ID:** 2097－0137(2025)04－0102－07

Let y be any real number. Define the binomial coefficient

$$\binom{y}{k}=\frac{y(y-1)(y-2)\cdots(y-k+1)}{k!}.$$

For any $n\in\mathbb{N}$, the central trinomial coefficients are given by

$$T_n:=[x^n](1+x+x^2)^n=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}\binom{2k}{k}.$$

It is known that T_n has many combinatorial interpretations (http://oeis.org). For example, T_n counts lattice paths running from $(0,0)$ to $(n,0)$ with steps $(1,1)$, $(1,-1)$ and $(1,0)$. By the formulae (3. 136) and (3. 137) in Gould (1972), the numbers T_n can also be written as

$$T_n=3^n\sum_{k=0}^{n}(-3)^{-k}\binom{n}{k}\binom{2k}{k}. \tag{1}$$

Sun (2014) proved that, for any prime $p>3$,

$$\sum_{k=0}^{p-1}12^{-k}\binom{2k}{k}T_k\equiv\left(\frac{6}{p}\right)\sum_{k=0}^{p-1}64^{-k}\binom{4k}{2k}\binom{2k}{k}\equiv\left(\frac{p}{3}\right)\pmod{p}, \tag{2}$$

where $\left(\frac{\cdot}{\cdot}\right)$ denotes the Legendre symbol. Recently, using combinatorial identities, Wang et al. (2024) obtained the following supercongruence: for any prime $p>3$,

$$\sum_{k=0}^{p-1}12^{-k}\binom{2k}{k}T_k\equiv\left(\frac{p}{3}\right)\frac{3^{p-1}+3}{4}\pmod{p^2}, \tag{3}$$

which is clearly an extension of (2). In this paper, we shall prove the following general result on T_n, which was proposed by Sun (2019).

Theorem 1 Let $p>3$ be a prime and let m be a positive integer with $p\nmid m$. Then, for any positive inte-

* **Received**: 2024－01－26 **Accepted**: 2024－10－12 **Published online**: 2025－04－21
Supported by National Natural Science Foundation of China (11971222, 12071208); Jiangsu Qinglan Project; Project of Guangzhou Huashang College (2022HSDS27)

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ZR20240034

ger r,

$$\left(mp^r\binom{2mp^{r-1}}{mp^{r-1}}\right)^{-1}\left(\sum_{k=0}^{mp^r-1}12^{-k}\binom{2k}{k}T_k-\left(\frac{p}{3}\right)\sum_{k=0}^{mp^{r-1}-1}12^{-k}\binom{2k}{k}T_k\right)\equiv\left(\frac{p}{3}\right)\frac{q_p(3)}{8}\cdot\frac{T_{mp^{r-1}-1}}{12^{mp^{r-1}-1}}\pmod{p}. \tag{4}$$

It is easy to see that when $m=r=1$, the supercongruence (4) reduces to (3). We mention that the second author (2021) has already proved that the left-hand side of (4) with $m=1$ is a p-adic integer.

1 Some lemmas

In order to prove Theorem 1, we need to establish the following lemmas.

Lemma 1(Beukers,1985) Let n be a nonnegative integer and let p be a prime. Let k, r be positive integers. Then

$$\binom{p^rn-1}{k}\equiv\binom{p^{r-1}n-1}{\left\lfloor\frac{k}{p}\right\rfloor}(-1)^{k-\left\lfloor\frac{k}{p}\right\rfloor}\left(1-np^r\sum_{j=1,p\nmid j}^{k}\frac{1}{j}\right)\pmod{p^{2r}}.$$

Lemma 2(Jacobsthal's binomial congruence) Let p be a prime. Then, for any nonnegative integers a, b and positive integers r, s, we have

$$\binom{p^ra}{p^sb}\equiv\binom{p^{r-1}a}{p^{s-1}b}(-1)^{p^sb-p^{s-1}b}\pmod{p^{2r+\min\{r,s\}-\delta_{p,3}-2\delta_{p,2}}}. \tag{5}$$

For $p\geqslant 5$, the congruence (5) was also confirmed by Gessel (1983) and Granville (1997), respectively.

Lemma 3(Osburn et al. , 2016) Let p be a prime and let n an integer such that $(p-1)\nmid n$. Then, for all integers $r\geqslant 0$,

$$\sum_{k=1,p\nmid k}^{p^r-1}k^n\equiv 0\pmod{p^r}.$$

If, additionally, n is even, then, for primes $p>5$,

$$\sum_{k=1,p\nmid k}^{\frac{p^r-1}{2}}\frac{1}{k^n}\equiv 0\pmod{p^r}. \tag{6}$$

It is clear from (6) that

$$\sum_{k=1,p\nmid k}^{p^r-1}\frac{1}{k^n}=\sum_{k=1,p\nmid k}^{\frac{p^r-1}{2}}\left(\frac{1}{k^n}+\frac{1}{(p^r-k)^n}\right)\equiv 0\pmod{p^r}.$$

Lemma 4(supercongruence) Let $p>3$ be a prime and let m be a positive integer such that $p\nmid m$. Let l be a nonnegative integer and $s\in\mathbb{Z}^+$. For any positive integer r, we have

$$\prod_{j=1,p\nmid j}^{mp^r-1}\left(1-\frac{2mp^r}{j}\right)\prod_{j=1,p\nmid j}^{p^{s+1}l+\frac{p^{s+1}-1}{2}}\left(1-\frac{mp^r}{j}\right)\equiv 4^{mp^r-mp^{r-1}}\pmod{p^{s+2}}. \tag{7}$$

Proof Clearly for any positive integer l, we get

$$\sum_{j=1,p\nmid j}^{p^{s+1}l-1}\frac{1}{j}=\sum_{j=0}^{l-1}\sum_{\left\lfloor\frac{k}{p^{s+1}}\right\rfloor=j,p\nmid k}\frac{1}{k}\equiv 0\pmod{p^{2s+2}} \tag{8}$$

and

$$\sum_{j=1,p\nmid j}^{p^{s+1}l-1}\frac{1}{j^2}=\sum_{j=0}^{l-1}\sum_{\left\lfloor\frac{k}{p^{s+1}}\right\rfloor=j,p\nmid k}\frac{1}{k^2}\equiv 0\pmod{p^{s+1}}.$$

It follows that

$$\begin{aligned}\prod_{j=1,p\nmid j}^{p^{s+1}l+\frac{p^{s+1}-1}{2}}\left(1-\frac{mp^r}{j}\right)&=\prod_{j=1,p\nmid j}^{p^{s+1}l-1}\left(1-\frac{mp^r}{j}\right)\prod_{j=1,p\nmid j}^{\frac{p^{s+1}-1}{2}}\left(1-\frac{mp^r}{p^{s+1}l+j}\right)\\&\equiv\left(1-\sum_{j=1,p\nmid j}^{p^{s+1}l-1}\frac{mp^r}{j}+\frac{m^2p^{2r}}{2}\left(\left(\sum_{j=1,p\nmid j}^{p^{s+1}l-1}\frac{1}{j}\right)^2-\sum_{j=1,p\nmid j}^{p^{s+1}l-1}\frac{1}{j^2}\right)\right)\prod_{j=1,p\nmid j}^{\frac{p^{s+1}-1}{2}}\left(1-\frac{mp^r}{j}\right)\\&\equiv\prod_{j=1,p\nmid j}^{\frac{p^{s+1}-1}{2}}\left(1-\frac{mp^r}{j}\right)=\frac{\left(\frac{-1}{p}\right)\binom{mp^r-1}{\frac{p^{s+1}-1}{2}}}{\binom{mp^{r-1}-1}{\frac{p^s-1}{2}}}\pmod{p^{s+2}}.\end{aligned}\tag{9}$$

By (9), the facts $\binom{n}{k}\binom{k}{l}=\binom{n}{l}\binom{n-l}{k-l}$ and $\binom{2k}{k}=\binom{-\frac{1}{2}}{k}(-4)^k$, we obtain

$$\begin{aligned}&\prod_{j=1,p\nmid j}^{mp^r-1}\left(1-\frac{2mp^r}{j}\right)\prod_{j=1,p\nmid j}^{p^{s+1}l+\frac{p^{s+1}-1}{2}}\left(1-\frac{mp^r}{j}\right)\\&\equiv\frac{\left(\frac{-1}{p}\right)\binom{mp^r-1}{\frac{p^{s+1}-1}{2}}\binom{2mp^r}{mp^r}}{\binom{mp^{r-1}-1}{\frac{p^s-1}{2}}\binom{2mp^{r-1}}{mp^{r-1}}}\\&=\frac{\left(\frac{-1}{p}\right)\binom{-\frac{1}{2}}{mp^r}4^{mp^r-mp^{r-1}}\binom{mp^r}{\frac{p^{s+1}-1}{2}}\left(mp^r-\frac{p^{s+1}-1}{2}\right)}{\binom{-\frac{1}{2}}{mp^{r-1}}\binom{mp^{r-1}}{\frac{p^s-1}{2}}\left(mp^{r-1}-\frac{p^s-1}{2}\right)p}\\&=\frac{\left(\frac{-1}{p}\right)\binom{-\frac{1}{2}}{\frac{p^{s+1}-1}{2}}4^{mp^r-mp^{r-1}}\binom{-\frac{p^{s+1}}{2}-1}{mp^r-\frac{p^{s+1}-1}{2}-1}}{\binom{-\frac{1}{2}}{\frac{p^s-1}{2}}\binom{-\frac{p^s}{2}-1}{mp^{r-1}-\frac{p^s-1}{2}-1}}\pmod{p^{s+2}}.\end{aligned}\tag{10}$$

Observe that

$$H_{\frac{p-1}{2}}\equiv-2q_p(2)\pmod{p}.\tag{11}$$

With the help of (8) and (11), we have

$$\frac{\left(\frac{-1}{p}\right)\binom{-\frac{p^{s+1}}{2}-1}{mp^{r}-\frac{p^{s+1}-1}{2}-1}}{\binom{-\frac{p^{s}}{2}-1}{mp^{r-1}-\frac{p^{s}-1}{2}-1}}=\prod_{j=1,p\nmid j}^{mp^{r}-\frac{p^{s+1}-1}{2}-1}\frac{\frac{p^{s+1}}{2}+j}{j}\equiv\left(1+\sum_{j=1,p\nmid j}^{p\left(mp^{r-1}-\frac{p^{s}+1}{2}\right)-1}\frac{p^{s+1}}{2j}\right)\prod_{j=1}^{\frac{p-1}{2}}\left(1+\frac{p^{s+1}}{2\left(mp^{r}-\frac{p^{s+1}+p}{2}+j\right)}\right)$$

$$\equiv 1+\frac{p^{s+1}}{2}H_{\frac{p-1}{2}}\equiv 1-p^{s+1}q_{p}(2)\pmod{p^{s+2}} \tag{12}$$

and

$$\frac{\binom{-\frac{1}{2}}{\frac{p^{s+1}-1}{2}}}{\binom{-\frac{1}{2}}{\frac{p^{s}-1}{2}}}=\frac{\binom{p^{s+1}-1}{\frac{p^{s+1}-1}{2}}}{\binom{p^{s}-1}{\frac{p^{s}-1}{2}}(-4)^{\frac{p^{s+1}-p^{s}}{2}}}=\frac{\prod_{j=1,p\nmid j}^{\frac{p^{s+1}-p}{2}-1}\left(1-\frac{p^{s+1}}{j}\right)\prod_{j=1}^{\frac{p-1}{2}}\left(1-\frac{p^{s+1}}{\frac{p^{s+1}-p}{2}+j}\right)}{2^{p^{s+1}-p^{s}}}$$

$$\equiv\frac{1-p^{s+1}H_{\frac{p-1}{2}}}{\left(1+pq_{p}(2)\right)^{p^{s}}}\equiv 1+p^{s+1}q_{p}(2)\pmod{p^{s+2}}. \tag{13}$$

The congruence (7) then easily follows from (10) and (12)-(13).

2 Proof of Theorem 1

Wang et al. (2024) showed that

$$\sum_{k=0}^{n-1}12^{-k}\binom{2k}{k}T_{k}=\binom{2n}{n}\frac{2n}{4^{n}}\sum_{k=0}^{n-1}\frac{1}{2k+1}\left(-\frac{1}{3}\right)^{k}\binom{n-1}{k}\binom{2k}{k}. \tag{14}$$

Let $p\geqslant 5$ be a prime and $m\in\mathbb{Z}^{+}$ with $p\nmid m$. Taking $n=mp^{r-1+j}$ for $j\in\{0,1\}$ and $r\in\mathbb{Z}^{+}$ in (14) yields

$$\left(mp^{r-1}\binom{2mp^{r-1}}{mp^{r-1}}\right)^{-1}\left(\sum_{k=0}^{mp^{r}-1}12^{-k}\binom{2k}{k}T_{k}-\left(\frac{p}{3}\right)\sum_{k=0}^{mp^{r-1}-1}12^{-k}\binom{2k}{k}T_{k}\right)$$

$$=\binom{2mp^{r-1}}{mp^{r-1}}^{-1}\left(\binom{2mp^{r}}{mp^{r}}\frac{2p}{4^{mp^{r}}}\sum_{k=0}^{mp^{r}-1}\frac{1}{2k+1}\left(-\frac{1}{3}\right)^{k}\binom{mp^{r}-1}{k}\binom{2k}{k}\right.$$

$$\left.-\left(\frac{p}{3}\right)\binom{2mp^{r-1}}{mp^{r-1}}\frac{2}{4^{mp^{r-1}}}\sum_{k=0}^{mp^{r-1}-1}\frac{1}{2k+1}\left(-\frac{1}{3}\right)^{k}\binom{mp^{r-1}-1}{k}\binom{2k}{k}\right). \tag{15}$$

In light of Lemma 2. 2 (Zhang, 2021) and the fact that $3^{\frac{p-1}{2}}\equiv\left(\frac{3}{p}\right)\pmod{p}$, we have

$$\sum_{l=0}^{\frac{p-3}{2}}\frac{1}{(2l+1)3^{l}}\binom{2l}{l}\equiv\frac{1}{p}\left(\frac{-1}{p}\right)\frac{1+3^{p}-4^{p}}{4\cdot 3^{\frac{p-1}{2}}}=\frac{1}{p}\left(\frac{-1}{p}\right)\frac{1+3\left(1+pq_{p}(3)\right)-4\left(1+pq_{p}(2)\right)^{2}}{4\cdot 3^{\frac{p-1}{2}}}$$

$$\equiv\left(\frac{p}{3}\right)\frac{3q_{p}(3)-8q_{p}(2)}{4}\pmod{p}. \tag{16}$$

By Lemma 1 and Lemma 2, Lucas' theorem, (1) and (16), we obtain

$$\frac{2p\binom{2mp^r}{mp^r}}{4^{mp^r}}\binom{2mp^{r-1}}{mp^{r-1}}^{-1}\sum_{k=0,\,p\nmid(2k+1)}^{mp^r-1}\binom{mp^r-1}{k}\binom{2k}{k}\frac{\left(-\frac{1}{3}\right)^k}{2k+1}\equiv\frac{2p}{4^{mp^r}}\sum_{k=0}^{mp^{r-1}-1}\sum_{l=0,\,l\neq\frac{p-1}{2}}^{p-1}\binom{mp^r-1}{pk+l}\binom{2pk+2l}{pk+l}\frac{\left(-\frac{1}{3}\right)^{pk+l}}{2pk+2l+1}$$

$$\equiv\frac{2p}{4^{mp^r}}\sum_{k=0}^{mp^{r-1}-1}\binom{mp^{r-1}-1}{k}\binom{2k}{k}\left(-\frac{1}{3}\right)^k\sum_{l=0}^{\frac{p-3}{2}}\frac{\binom{2l}{l}}{(2l+1)3^l}$$

$$\equiv\frac{T_{mp^{r-1}-1}}{12^{mp^{r-1}-1}}\cdot\frac{3q_p(3)-8q_p(2)}{8}\cdot\left(\frac{p}{3}\right)p\pmod{p^2},\tag{17}$$

where we have utilized the fact $\binom{2l}{l}\equiv 0\pmod p$ for $l\in\left\{\frac{p+1}{2},\cdots,p-1\right\}$. For any positive integer s, recall that

$$\binom{2mp^r}{mp^r}=2\prod_{j=1}^{mp^r-1}\frac{2mp^r-j}{j}=\binom{2mp^{r-1}}{mp^{r-1}}\prod_{j=1,\,p\nmid j}^{mp^r-1}\left(1-\frac{2mp^r}{j}\right),\tag{18}$$

and

$$\binom{mp^r-1}{p^{s+1}l+\frac{p^{s+1}-1}{2}}=\left(\frac{-1}{p}\right)\binom{mp^{r-1}-1}{p^sl+\frac{p^s-1}{2}}\prod_{j=1,\,p\nmid j}^{p^{s+1}l+\frac{p^{s+1}-1}{2}}\left(1-\frac{mp^r}{j}\right).\tag{19}$$

Since, for any positive integer s,

$$3^{\frac{p^s-p^{s-1}}{2}}\equiv\left(\frac{3}{p}\right)\pmod{p^s},$$

we get

$$3^{\frac{p-1}{2}}\left(\frac{3}{p}\right)\equiv\frac{\left(3^{\frac{p-1}{2}}-\left(\frac{3}{p}\right)\right)^2}{2}+\left(\frac{3}{p}\right)3^{\frac{p-1}{2}}=\frac{3^{p-1}+1}{2}=1+\frac{pq_p(3)}{2}\pmod{p^2}.$$

It follows that

$$\left(\frac{-1}{p}\right)\frac{1}{3^{\frac{p^s(p-1)}{2}}}\equiv\left(\frac{p}{3}\right)\frac{1}{1+\frac{p^{s+1}q_p(3)}{2}}\equiv\left(\frac{p}{3}\right)\left(1-\frac{p^{s+1}q_p(3)}{2}\right)\pmod{p^{s+2}}.\tag{20}$$

Note that

$$\left(-\frac{1}{3}\right)^{p^{s+1}l-p^sl}=\frac{1}{\left(1+pq_p(3)\right)^{p^sl}}\equiv\frac{1}{1+p^{s+1}lq_p(3)}\equiv 1-p^{s+1}lq_p(3)\pmod{p^{s+2}}.\tag{21}$$

With the help of (8) and (11), we get

$$\binom{2p^{s+1}l+p^{s+1}-1}{p^{s+1}l+\frac{p^{s+1}-1}{2}}=\left(\frac{-1}{p}\right)\binom{2p^sl+p^s-1}{p^sl+\frac{p^s-1}{2}}\prod_{j=1,\,p\nmid j}^{p^{s+1}l+\frac{p^{s+1}-1}{2}}\left(1-\frac{p^{s+1}(2l+1)}{j}\right)$$

$$\equiv\left(\frac{-1}{p}\right)\binom{2p^sl+p^s-1}{p^sl+\frac{p^s-1}{2}}\left(1-\sum_{j=1,\,p\nmid j}^{p^{s+1}l+\frac{p^{s+1}-1}{2}}\frac{p^{s+1}(2l+1)}{j}\right)\equiv\left(\frac{-1}{p}\right)\binom{2p^sl+p^s-1}{p^sl+\frac{p^s-1}{2}}\left(1-\sum_{j=1}^{\frac{p-1}{2}}\frac{p^{s+1}(2l+1)}{p^{s+1}l+\frac{p^{s+1}-p}{2}+j}\right)$$

$$\equiv\left(\frac{-1}{p}\right)\binom{2p^sl+p^s-1}{p^sl+\frac{p^s-1}{2}}\left(1+2p^{s+1}(2l+1)q_p(2)\right)\pmod{p^{s+2}}.\tag{22}$$

Combining (15), (18) – (22) and Lemma 4, we arrive at

$$\frac{1}{\binom{2mp^{r-1}}{mp^{r-1}}}\left\{\frac{2}{4^{mp^r}p^s}\binom{2mp^r}{mp^r}\sum_{l=0,p\nmid(2l+1)}\binom{mp^r-1}{p^{s+1}l+\frac{p^{s+1}-1}{2}}\binom{2p^{s+1}l+p^{s+1}-1}{p^{s+1}l+\frac{p^{s+1}-1}{2}}\frac{\left(-\frac{1}{3}\right)^{p^{s+1}l+\frac{p^{s+1}-1}{2}}}{2l+1}\right.$$

$$\left.-\left(\frac{p}{3}\right)\cdot\frac{2\binom{2mp^{r-1}}{mp^{r-1}}}{4^{mp^{r-1}}p^s}\sum_{l=0,p\nmid(2l+1)}\binom{mp^{r-1}-1}{p^sl+\frac{p^s-1}{2}}\binom{2p^sl+p^s-1}{p^sl+\frac{p^s-1}{2}}\frac{\left(-\frac{1}{3}\right)^{p^sl+\frac{p^s-1}{2}}}{2l+1}\right\}$$

$$\equiv\left(\frac{p}{3}\right)\frac{2}{4^{mp^r}p^s}\sum_{l=0,p\nmid(2l+1)}\binom{mp^{r-1}-1}{p^sl+\frac{p^s-1}{2}}\binom{2p^sl+p^s-1}{p^sl+\frac{p^s-1}{2}}\frac{\left(-\frac{1}{3}\right)^{p^sl+\frac{p^s-1}{2}}}{2l+1}$$

$$\times\left\{\prod_{j=1,p\nmid j}^{mp^r-1}\left(1-\frac{2mp^r}{j}\right)\prod_{j=1,p\nmid j}^{p^{s+1}l+\frac{p^{s+1}-1}{2}}\left(1-\frac{mp^r}{j}\right)\left(1+2p^{s+1}(2l+1)q_p(2)\right)\left(\frac{p}{3}\right)\left(-\frac{1}{3}\right)^{p^{s+1}l-p^sl+\frac{p^{s+1}-p^s}{2}}-4^{mp^{r-1}(p-1)}\right\}$$

$$\equiv\frac{p\left(4q_p(2)-q_p(3)\right)}{\left(\frac{p}{3}\right)4^{mp^{r-1}}}\sum_{l=0,p\nmid(2l+1)}\binom{mp^{r-1}-1}{p^sl+\frac{p^s-1}{2}}\binom{2p^sl+p^s-1}{p^sl+\frac{p^s-1}{2}}\left(-\frac{1}{3}\right)^{p^sl+\frac{p^s-1}{2}}\pmod{p^2}.\tag{23}$$

Suppose that $m=\sum_{k=0}^{a}m_kp^k$ with $m_0,m_a\in\{1,\cdots,p-1\}$ and $m_k\in\{0,\cdots,p-1\}$ for $k\in\{1,\cdots,a-1\}$. Substituting (17) and (23) into (15), we can prove by induction that, for any positive integer s,

$$\frac{1}{mp^{r-1}\binom{2mp^{r-1}}{mp^{r-1}}}\left(\sum_{k=0}^{mp^r-1}\frac{\binom{2k}{k}}{12^k}T_k-\left(\frac{p}{3}\right)\sum_{k=0}^{mp^{r-1}-1}\frac{\binom{2k}{k}}{12^k}T_k\right)$$

$$\equiv\frac{T_{mp^{r-1}-1}}{12^{mp^{r-1}-1}}\cdot\frac{3q_p(3)-8q_p(2)}{8}\cdot\left(\frac{p}{3}\right)p+\frac{1}{\binom{2mp^{r-1}}{mp^{r-1}}}\sum_{s=0}^{a+r-1}\left\{\frac{2\binom{2mp^r}{mp^r}}{4^{mp^r}p^s}\right.$$

$$\cdot\sum_{l=0,p\nmid(2l+1)}\binom{mp^r-1}{p^{s+1}l+\frac{p^{s+1}-1}{2}}\binom{2p^{s+1}l+p^{s+1}-1}{p^{s+1}l+\frac{p^{s+1}-1}{2}}\frac{\left(-\frac{1}{3}\right)^{p^{s+1}l+\frac{p^{s+1}-1}{2}}}{2l+1}$$

$$\left.-\left(\frac{p}{3}\right)\cdot\frac{2\binom{2mp^{r-1}}{mp^{r-1}}}{4^{mp^{r-1}}p^s}\sum_{l=0,p\nmid(2l+1)}\binom{mp^{r-1}-1}{p^sl+\frac{p^s-1}{2}}\binom{2p^sl+p^s-1}{p^sl+\frac{p^s-1}{2}}\frac{\left(-\frac{1}{3}\right)^{p^sl+\frac{p^s-1}{2}}}{2l+1}\right\}\pmod{p^2}.\tag{24}$$

Noting that

$$\frac{p\left(4q_p(2)-q_p(3)\right)}{\left(\frac{p}{3}\right)4^{mp^{r-1}}}\sum_{s=0}^{a+r-1}\sum_{l=0,p\nmid(2l+1)}\binom{mp^{r-1}-1}{p^s l+\frac{p^s-1}{2}}\binom{2p^s l+p^s-1}{p^s l+\frac{p^s-1}{2}}\left(-\frac{1}{3}\right)^{p^s l+\frac{p^s-1}{2}}$$

$$=\frac{p\left(4q_p(2)-q_p(3)\right)}{\left(\frac{p}{3}\right)4^{mp^{r-1}}}\sum_{k=0}^{mp^{r-1}-1}\binom{mp^{r-1}-1}{k}\binom{2k}{k}\left(-\frac{1}{3}\right)^k=\frac{T_{mp^{r-1}-1}}{12^{mp^{r-1}-1}}\frac{4q_p(2)-q_p(3)}{4}\cdot\left(\frac{p}{3}\right)p. \tag{25}$$

By (1) and substituting (23) and (25) into (24), we arrive at (4). This concludes our proof of Theorem 1.

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一个与中心三项式系数有关的同余式

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摘　要：定义中心三项式系数 T_n 为 $\left(1+x+x^2\right)^n$ 展开式中 x^n 项的系数. 我们证明了孙智伟提出的关于中心三项式系数与中心二项式系数和式的同余式猜想.

关键词：超同余式；中心三项式系数；中心二项式系数；Fermat商；Legendre符号

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