中山大学学报　1964年　第4期

常系数二阶两个自变数两个未知函数綫性抛物型偏微分方程組

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§1 前　言

命A,B,C代表三个二行二列的实数方陣，矩陣形式的偏微分方程

$$A\frac{\partial^2}{\partial x^2}\binom{u}{v}+2B\frac{\partial^2}{\partial x\partial y}\binom{u}{v}+C\frac{\partial^2}{\partial y^2}\binom{u}{v}=0 \tag{1.1}$$

实际上是两个自变数 x,y，两个未知函数 u,v 的两个方程所成的方程組。

行列式

$$Q(\xi,\eta)=|A\xi^2+2B\xi\eta+C\eta^2| \tag{1.2}$$

定义为微分方程組（1,1）的特征四次型，如果特征四次型的根是一个实的四重根，則方程組(1,1)是抛物型的，

方程組(1,1)与方程組

$$A_1\frac{\partial^2}{\partial x^2}\binom{u}{v}+2B_1\frac{\partial^2}{\partial x\partial y}\binom{u}{v}+C_1\frac{\partial^2}{\partial y^2}\binom{u}{v}=0 \tag{1.3}$$

的等价关系是由以下三种运算来定义的：

i)方程之間的綫性組合：即有一个二行二列的滿秩方陣P左乘方程組(1,1)，使

$A_1=PA$，$B_1=PB$，$C_1=PC$

ii)未知函数的綫性变换：即是有一个二行二列的滿秩方陣Q，当作未知函数的綫性变换

$$\binom{u}{v}=Q\binom{u_1}{v_1}$$

时，方程組(1,1)的系数所成的方陣 A,B,C 各变成

$$A_1 = AQ, \quad B_1 = BQ, \quad C_1 = CQ$$

iii)自变数的綫性变换：当自变数作綫性变换

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{vmatrix} p & q \\ r & s \end{vmatrix} \neq 0$$

时，方程組(1,1)的系数所成的方陣 A,B,C 各变成

$$A_1 = p^2A + 2pqB + q^2C,$$

$$B_1 = prA + 2(ps + qr)B + qsC,$$

$$C_1 = r^2A + 2rsB + s^2C,$$

若方程組(1,1)可以連續运用以上三种运算而成为方程組(1,3)，則称它們是互相等价的，若特征方程(1,2)的根是一实四重根时，經过上述的 i)，ii)，iii)三种变换可以化成标准型〔1〕：

$$(\text{I}) \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial^2}{\partial x^2} \begin{pmatrix} u \\ v \end{pmatrix} + 2 \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix} \frac{\partial^2}{\partial x \partial y} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 4b \end{pmatrix} \frac{\partial^2}{\partial y^2} \begin{pmatrix} u \\ v \end{pmatrix} = 0, \quad b \neq 0.$$

§2 方程組(I)的一般解

命

$$\frac{\partial u}{\partial x} = p, \quad \frac{\partial u}{\partial y} = q, \quad \frac{\partial v}{\partial x} = r, \quad \frac{\partial v}{\partial y} = s$$

則(I)可以写成

$$\frac{\partial}{\partial x} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} + \begin{pmatrix} 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 \\ 2b & 0 & 0 & 4b \\ 0 & 0 & -1 & 0 \end{pmatrix} \frac{\partial}{\partial y} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = 0 \qquad (2.1)$$

以

$$P = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & \frac{1}{2b} & 0 \\ \frac{1}{4b} & 0 & 0 & 0 \\ 0 & -\frac{1}{4b} & 0 & 0 \end{pmatrix}$$

左乘(2,1)并且命

$$\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = P^{-1} \begin{pmatrix} p_1 \\ q_1 \\ r_1 \\ s_1 \end{pmatrix}$$

則(2,1)可以变成

$$\frac{\partial}{\partial x}\begin{pmatrix} p_1 \\ q_1 \\ r_1 \\ s_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \frac{\partial}{\partial y}\begin{pmatrix} p_1 \\ q_1 \\ r_1 \\ s_1 \end{pmatrix} = 0 \qquad (2.2)$$

由(2.2)可以求得

$$\begin{pmatrix} p_1 \\ q_1 \\ r_1 \\ s_1 \end{pmatrix}$$

的表达式，从而可以求得方程組(Ⅰ)的一般解是

$$(\text{Ⅱ}) \quad u = \frac{1}{3!}x^3 f_1''(y) - \frac{1}{2}x^2 f_2'(y) + x f_3(y) + f_4(y)$$

$$(\text{Ⅲ}) \quad v = \frac{1}{8b}\int_0^y f_1(t)dt - \frac{x^2}{4} f_1'(y) + \frac{x}{2} f_2(y) - \frac{1}{2}\int_0^y f_3(t)dt.$$

§3 Cauchy 問題

Cauchy 問題是求方程組(Ⅰ)的解 $\begin{pmatrix} u \\ v \end{pmatrix}$，使它們适合于

$$u\Big|_{y=cx} = \varphi_1(x), \quad u\Big|_{y=cx} = \varphi_2(x), \quad c \neq 0, \qquad (3.1)$$

$$\frac{\partial u}{\partial n}\Big|_{y=cx} = \psi_1(x), \quad \frac{\partial u}{\partial n}\Big|_{y=cx} = \psi_2(x), \quad -\infty < x < +\infty,$$

这儿

$$\frac{\partial}{\partial n} = \frac{1}{\sqrt{1+c^2}}\left(-c\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)$$

φ_i, ψ_i 是已知的函数，φ_i 是三阶連續可微的，ψ_i 是二阶連續可微的函数。

由(Ⅱ)，(Ⅲ)及(3.1)有

$$\frac{x^3}{3!} f_1''(cx) - \frac{x^2}{2!} f_2'(cx) + x f_3(cx) + f_4(cx) = \varphi_1(x) \qquad (3.2)$$

$$\frac{1}{8b}\int_0^{cx} f_1(t)dt - \frac{x^2}{4}f_1'(cx) + \frac{x}{2}f_2(cx) - \frac{1}{2}\int_0^{cx} f_3(t)dt = \varphi_2(x) \quad (3.3)$$

$$\frac{1}{\sqrt{1+c^2}}\Big[-c\Big(\frac{x^2}{2}f_1''(cx) - xf_2'(cx) + f_3(cx)\Big) + \frac{x^3}{3!}f_1'''(cx) - \\ - \frac{x^2}{2}f_2''(cx) + xf_3'(cx) + f_4'(cx)\Big] = \psi_1(x) \quad (3.4)$$

$$\frac{1}{\sqrt{1+c^2}}\Big[-c\Big(-\frac{x}{2}f_1'(cx) + \frac{1}{2}f_2(cx)\Big) + \frac{1}{8b}f_1(cx) - \\ - \frac{x^2}{4}f_1''(cx) + \frac{x^2}{2}f_2'(cx) - \frac{1}{2}f_3(cx)\Big] = \psi_2(x) \quad (3.5)$$

(3.2)对于 x 求导数减去(3.4)乘以 $c\sqrt{1+c^2}$ 得

$$(1+c^2)\Big\{\frac{x^2}{2}f_1''(cx) - xf_2'(cx) + f_3(cx)\Big\} = \varphi_1'(x) - c\sqrt{1+c^2}\,\psi_1(x). \quad (3.6)$$

(3.3)对 x 求导数，然后再乘以$(1+c^2)$加上(3.6)乘以 $\frac{c}{2}$ 得.

$$(1+c^2)\Big(-\frac{x}{2}f_1'(cx) + \frac{1}{2}f_2(cx) + \frac{c}{8b}f_1(cx)\Big) = \\ = (1+c^2)\varphi_2'(x) + \frac{c}{2}\Big(\varphi_1'(x) - c\sqrt{1+c^2}\,\psi_1(x)\Big), \quad (3.7)$$

(3.5) 乘 以 $(1+c^2)^{\frac{3}{2}}$ 加上(3.6)乘以$\frac{1}{2}$得，

$$-(1+c^2)\Big[c\Big(\frac{x}{2}f_1'(cx) + \frac{1}{2}f_2(cx)\Big) - \frac{1}{8b}f_1(cx)\Big] = \quad (3.8) \\ = (1+c^2)\sqrt{1+c^2}\,\psi_2(x) + \frac{1}{2}(\varphi_1'(x) - c\sqrt{1+c^2}\,\psi_1(x))$$

(3.7)乘以 c 加上(3.8)得，

$$f_1(x) = \frac{8b}{1+c^2}\Big[\frac{1}{2}\varphi_1'\Big(\frac{x}{c}\Big) + c\varphi_2'\Big(\frac{x}{c}\Big) - \frac{c\sqrt{1+c^2}}{2}\psi_1\Big(\frac{x}{c}\Big) + \\ + \sqrt{1+c^2}\,\psi_2\Big(\frac{x}{2}\Big)\Big] \quad (3.9)$$

由(3.8)有

$$f_2(x) = \frac{x}{c}f_1'(x) + \frac{1}{4bc}f_1(x) - \frac{2}{(1+c^2)c}\Big[(1+c^2)\sqrt{1+c^2}\,\psi_2\Big(\frac{x}{c}\Big) + \\ + \frac{1}{2}\Big(\varphi_1'\Big(\frac{x}{c}\Big) - c\sqrt{1+c^2}\,\psi_1\Big(\frac{x}{c}\Big)\Big)\Big].$$

由(3.3)有

$$-\frac{1}{2}\int_0^x f_3(t)dt=\varphi_2\left(\frac{x}{c}\right)-\frac{1}{8b}\int_0^x f_1(t)dt+\frac{x^2}{4c^2}f_1'(x)-\frac{x}{2c}f_2(x)$$

再由(3.2)可得$f_4(x)$由此立得 Cauchy 問題的解是：

$$\begin{aligned}
u=&\frac{8b}{1+c^2}\left\{\left(\frac{1}{3!}\left(x-\frac{y}{c}\right)^3\frac{d^2}{dy^2}\right)-\left(\frac{1}{2c}\left(1+\frac{1}{4b}\right)\left(x-\frac{y}{c}\right)^2\frac{d}{dy}\right)+\right.\\
&\left.+\frac{1}{4b}\left(1+\frac{1}{c^2}\right)\left(x-\frac{y}{c}\right)\right\}\left[\frac{1}{2}\varphi_1'\left(\frac{y}{c}\right)+c\varphi_2'\left(\frac{y}{c}\right)-\right.\\
&\left.-\frac{c\sqrt{1+c^2}}{2}\psi_1\left(\frac{y}{c}\right)+\sqrt{1+c^2}\,\psi_2\left(\frac{y}{c}\right)\right]+\\
&+\frac{1}{(1+c^2)c}\left(x-\frac{y}{c}\right)^2\frac{d}{dy}\left[(1+c^2)^{-\frac{3}{2}}\psi_2\left(\frac{y}{c}\right)+\frac{1}{2}\left(\varphi_1'\left(\frac{y}{c}\right)-\right.\right.\\
&\left.\left.-c\sqrt{1+c^2}\,\psi_1\left(\frac{y}{c}\right)\right)\right]-\frac{2}{(1+c^2)c^2}\left(x-\frac{y}{c}\right)\left[(1+c^2)^{-\frac{3}{2}}\psi_2\left(\frac{y}{c}\right)+\right.\\
&\left.+\frac{1}{2}\left(\varphi_1'\left(\frac{y}{c}\right)-c\sqrt{1+c^2}\,\psi_1\left(\frac{y}{c}\right)\right)\right]-\frac{2}{c}\left(x-\frac{y}{c}\right)\varphi_2'\left(\frac{y}{c}\right)+\\
&+\varphi_1\left(\frac{y}{c}\right)
\end{aligned}$$

$$\begin{aligned}
v=&-\frac{2b}{c(1+c^2)}\left(x-\frac{y}{c}\right)^2\left[\frac{1}{2}\varphi_1''\left(\frac{y}{c}\right)+c\varphi_2''\left(\frac{y}{c}\right)-\right.\\
&\left.-\frac{c\sqrt{1+c^2}}{2}\psi_1'\left(\frac{y}{c}\right)+\sqrt{1+c^2}\,\psi_2'\left(\frac{y}{c}\right)\right]+\\
&+\frac{1}{(1+c^2)c}\left(x-\frac{y}{c}\right)\left[\frac{1}{2}\varphi_1'\left(\frac{y}{c}\right)+c\varphi_2'\left(\frac{y}{c}\right)-\right.\\
&\left.-\frac{c\sqrt{1+c^2}}{2}\psi_1\left(\frac{y}{c}\right)+\sqrt{1+c^2}\,\psi_2\left(\frac{y}{c}\right)\right]-\\
&-\frac{1}{(1+c^2)c}\left(x-\frac{y}{c}\right)\left((1+c^2)^{-\frac{3}{2}}\psi_2\left(\frac{y}{c}\right)+\frac{1}{2}\left(\varphi_1'\left(\frac{y}{c}\right)-\right.\right.\\
&\left.\left.-c\sqrt{1+c^2}\,\psi_1\left(\frac{y}{c}\right)\right)\right)+\varphi_2\left(\frac{y}{c}\right).
\end{aligned}$$

§4 第一問題

第一問題是求方程組(Ⅰ)的解 u 及 v，使它适合于

$$(4.1)\qquad \begin{aligned} u\big|_{y=\alpha x}&=\varphi_1(x) \qquad & u\big|_{y=\beta x}&=\varphi_2(x)\\ v\big|_{y=\alpha x}&=\psi_1(x) \qquad & v\big|_{y=\beta x}&=\psi_2(x)\end{aligned}\qquad (x\neq 0)$$

这儿$\alpha.\beta$是不为零的实数，$\alpha\neq\beta$，φ_i,ψ_i是已知的函数，φ_i,ψ_i是三阶可微的連續函数。

由(Ⅱ)，(Ⅲ)及(4.1)有

$$\frac{1}{3!\alpha^3}x^3f_1''(x)-\frac{1}{2\alpha^2}x^2f_2'(x)+\frac{1}{\alpha}xf_3(x)+f_4(x)=\varphi_1\left(\frac{x}{\alpha}\right) \quad (4.2)$$

$$\frac{1}{3!\beta^3}x^3f_1''(x)-\frac{1}{2\beta^2}x^2f_2'(x)+\frac{1}{\beta}xf_3(x)+f_4(x)=\varphi_2\left(\frac{x}{\beta}\right) \quad (4.3)$$

$$\frac{1}{8b}\int_0^x f_1(t)dt-\frac{x^2}{4\alpha^2}f_1'(x)+\frac{x}{2\alpha}f_2(x)-\frac{1}{2}\int_0^x f_3(t)dt=\psi_1\left(\frac{x}{\alpha}\right) \quad (4.4)$$

$$\frac{1}{8b}\int_0^x f_1(t)dt-\frac{x^2}{4\beta^2}f_1'(x)+\frac{x}{2\beta}f_2(x)-\frac{1}{2}\int_0^x f_3(t)dt=\psi_2\left(\frac{x}{\beta}\right) \quad (4.5)$$

(4.2)减去(4.3)得

$$\frac{1}{3!}\left(\frac{1}{\alpha^3}-\frac{1}{\beta^3}\right)x^3f_1''(x)-\frac{1}{2}\left(\frac{1}{\alpha^2}-\frac{1}{\beta^2}\right)x^2f_2'(x) \quad (4.6)$$

$$+\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)xf_3(x)=\varphi_1\left(\frac{x}{\alpha}\right)-\varphi_2\left(\frac{x}{\beta}\right)$$

(4.4)对 x 求导数再乘以 x 得

$$xf_3(x)=\frac{1}{4b}xf_1(x)-\frac{1}{\alpha^2}x^2f_1'(x)-\frac{x^3}{2\alpha^2}f_1''(x)+ \quad (4.7)$$

$$\frac{1}{\alpha}xf_2(x)+\frac{1}{\alpha}x^2f_2'(x)-\frac{2}{\alpha}x\psi_1'\left(\frac{x}{\alpha}\right)$$

将(4.7) $xf_3(x)$ 的表示式代入(4.6)得

$$\frac{1}{3!}\left(\frac{1}{\alpha^3}-\frac{1}{\beta^3}\right)x^3f_1''(x)-\frac{1}{2}\left(\frac{1}{\alpha^2}-\frac{1}{\beta^2}\right)x^2f_2'(x)+\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\times$$

$$\times\left[\frac{x}{4b}xf_1(x)-\frac{1}{\alpha^2}x^2f_1'(x)-\frac{x^3}{2\alpha^2}f_1''(x)+\frac{1}{\alpha}xf_2(x)+\right.$$

$$\left.+\frac{1}{\alpha}x^2f_2'(x)-\frac{2}{\alpha}x\psi_1'\left(\frac{x}{\alpha}\right)\right]=\varphi_1\left(\frac{x}{\alpha}\right)-\varphi_2\left(\frac{x}{\beta}\right) \quad (4.8)$$

(4.4)减去(4.5)得

$$xf_2(x)=\frac{\beta+\alpha}{2\alpha\beta}x^2f_1'(x)+\frac{2\alpha\beta}{\beta-\alpha}\psi_1\left(\frac{x}{\alpha}\right)-\frac{2\alpha\beta}{\beta-\alpha}\psi_2\left(\frac{x}{\beta}\right) \quad (4.9)$$

将它代入(4.8)得

$$\frac{1}{12}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)^3x^3f_1''(x)-\frac{1}{4}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)^3x^2f_1'(x)+\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\frac{1}{4b}xf_1(x)$$

$$=\varphi_1\left(\frac{x}{\alpha}\right)-\varphi_2\left(\frac{x}{\beta}\right)-\frac{\alpha+\beta}{\alpha\beta}\left[\psi_1\left(\frac{x}{\alpha}\right)-\psi_2\left(\frac{x}{\beta}\right)\right] \quad (4.10)$$

若（4.10）两端乘以$\frac{1}{x}$，则（4.10）可以变成为一个Euler方程·显然当$x\neq 0$时，这方程的解是存在的。

由（4.10）可以求得

$$f_1(x)=\omega(x)$$

这儿$\omega(x)$含有二个任意常数

$$f_2(x)=\frac{1}{x}\left\{\frac{\beta+\alpha}{2\alpha\beta}x^2 f_1'(x)+\frac{2\alpha\beta}{\beta-\alpha}\psi_1\left(\frac{x}{\alpha}\right)-\frac{2\alpha\beta}{\beta-\alpha}\psi_2\left(\frac{x}{\beta}\right)\right\}$$

$$f_3(x)=\frac{1}{4b}f_1(x)-\frac{1}{\alpha^2}xf_1'(x)\frac{x^2}{2\alpha^2}f_1''(x)+\frac{1}{\alpha}f_2(x)+\frac{1}{\alpha}xf_2'(x)$$
$$-\frac{2}{\alpha}\psi_1'\left(\frac{x}{\alpha}\right)$$

同时，由（4.2）$f_4(x)$可用$f_1(x)$，$f_2(x)$，$f_3(x)$表示

由此可以求得第一問題的解：当$y\neq 0$时

$$u=\frac{1}{3!}\left(x^3-\frac{y^3}{\alpha^3}\right)\omega''(y)+\frac{1}{2}\left(x^2-\frac{y^2}{\alpha^2}\right)\frac{1}{y^2}\left\{\frac{\beta+\alpha}{2\alpha\beta}y^2\omega'(y)+\right.$$
$$\left.+\frac{2\alpha\beta}{\beta-\alpha}\psi_1\left(\frac{y}{\alpha}\right)-\frac{2\alpha\beta}{\beta-\alpha}\psi_2\left(\frac{y}{\beta}\right)\right\}-\frac{1}{2}\left(x^2-\frac{y^2}{\alpha^2}\right)\frac{1}{y}\times$$
$$\times\left\{\frac{\beta+\alpha}{2\alpha\beta}\frac{d}{dy}\left(y^2\omega'(y)+\frac{2\beta}{\beta-\alpha}\psi_1'\left(\frac{y}{\alpha}\right)-\frac{2\alpha}{\beta-\alpha}\psi'_2\left(\frac{y}{\alpha}\right)\right\}+\right.$$
$$+\left(x-\frac{y}{\alpha}\right)\left\{\frac{1}{4b}\omega(y)-\frac{1}{\alpha^2}y\omega'(y)-\frac{y^2}{2\alpha^2}\omega''(y)+\right.$$
$$+\frac{\alpha+\beta}{2\alpha^2\beta}y\frac{d}{dy}[y^2\omega(y)]+\frac{2\beta}{\beta-\alpha}y\left[\frac{1}{\alpha}\psi_1'\left(\frac{y}{\alpha}\right)-\frac{1}{\beta}\psi_2'\left(\frac{y}{\beta}\right)\right]-$$
$$\left.-\frac{2}{\alpha}\psi_1'\left(\frac{y}{\alpha}\right)\right\}+\varphi_1\left(\frac{y}{\alpha}\right)$$

$$v=\frac{1}{4}\left(\frac{y^2}{\alpha^2}-x^2\right)\omega'(y)+\frac{1}{2}\left(x-\frac{y}{\alpha}\right)\frac{1}{y}\left\{\frac{\beta+\alpha}{2\alpha\beta}y^2\omega'(y)+\frac{2\alpha\beta}{\beta-\alpha}\psi_1\left(\frac{y}{\alpha}\right)\right.$$
$$\left.-\frac{2\alpha\beta}{\beta-\alpha}\psi_2\left(\frac{y}{\beta}\right)\right\}+\psi_1\left(\frac{y}{\alpha}\right)$$

附記：若边值条件（4.1）换为

$$\left.\begin{array}{ll} u\Big|_{y=\alpha x}=\varphi_1(x) & u\Big|_{y=\beta x}=\varphi_2(x) \\ v\Big|_{y=\alpha x}=\psi_1(x) & v\Big|_{y=\gamma x}=\psi_2(x) \end{array}\quad(-\infty<x<+\infty)\right\}\qquad(4.11)$$

或者是

$$\left.\begin{array}{ll} u\Big|_{y=\alpha x}=\varphi_1(x) & u\Big|_{y=\beta x}=\varphi_2(x) \\ v\Big|_{rx}=\psi_1(x) & v\Big|_{y=\delta x}=\psi_2(x) \end{array}\quad(-\infty<x<+\infty)\right\}\quad(4.12)$$

时，这样的定解問題仍然是存在唯一的，这儿$\varphi_i\psi_i$是三阶可微的函数，而且本身直至其二阶导数在零点处为零，α，β，γ，δ是互不相等且不为零的实数。

§5 第二問題

第二問題是求方程組（Ⅰ）的解u及v，使它們适合于

$$\left.\begin{array}{ll} v\Big|_{y=\alpha x}=\psi_1(x) & v\Big|_{y=\beta x}=\psi_2(x) \\ v\Big|_{y=rx}=\psi_3(x) & u\Big|_{y=\alpha x}=\varphi_1(x) \end{array}\quad(-\infty<x<+\infty)\right\}\quad(5.1)$$

这儿α，β，r是互不相等的非零的实数，ψ_i，φ_1是已知的，三阶連續可微的函数，而且
$\psi_i(0)=\psi_i'(0)=\psi_i''(0)=\varphi_1(0)=\varphi_1'(0)=\varphi_1''(0)=0$

由（Ⅱ），（Ⅲ）及（5.1）有

$$\frac{1}{8b}\int_0^x f_1(t)dt-\frac{x^2}{4\alpha^2}f_1'(x)+\frac{x}{2\alpha}f_2(x)-\frac{1}{2}\int_0^x f_3(t)dt=\psi_1\left(\frac{x}{\alpha}\right)\quad(5.2)$$

$$\frac{1}{8b}\int_0^x f_1(t)dt-\frac{x^2}{4\beta^2}f_1'(x)+\frac{x}{2\beta}f_2(x)-\frac{1}{2}\int_0^x f_3(t)dt=\psi_2\left(\frac{x}{\beta}\right)\quad(5.3)$$

$$\frac{1}{8b}\int_0^x f_1(t)dt-\frac{x^2}{4r^2}f_1'(x)+\frac{x}{2r}f_2(x)-\frac{1}{2}\int_0^x f_3(t)dt=\psi_3\left(\frac{x}{r}\right)\quad(5.4)$$

$$\frac{1}{3!\alpha^3}x^3f_1''(x)-\frac{x^2}{2\alpha^2}f_2'(x)+\frac{x}{\alpha}f_3(x)+f_4(x)=\varphi_1\left(\frac{x}{\alpha}\right)\quad(5.5)$$

由（5.2）－（5.3）及（5.3）－（5.4）得

$$-\frac{1}{4}\left(\frac{1}{\alpha^2}-\frac{1}{\beta^2}\right)x^2f_1'(x)+\frac{1}{2}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)xf_2(x)=\psi_1\left(\frac{x}{\alpha}\right)-\psi_1\left(\frac{x}{\beta}\right)\quad(5.6)$$

$$-\frac{1}{4}\left(\frac{1}{\beta^2}-\frac{1}{r^2}\right)x^2f_1'(x)+\frac{1}{2}\left(\frac{1}{\beta}-\frac{1}{r}\right)xf_2(x)=\psi_2\left(\frac{x}{\beta}\right)-\psi_3\left(\frac{x}{r}\right)\quad(5.7)$$

由（5.6），（5.7）消去$f_2(x)$得

$$x^2f_1'(x)=-\frac{4\alpha r}{r-\alpha}\left\{\frac{\alpha\beta}{\beta-\alpha}\left[\psi_1\left(\frac{x}{\alpha}\right)-\psi_2\left(\frac{x}{\beta}\right)\right]-\frac{\beta r}{\beta-r}\left[\psi_2\left(\frac{x}{\beta}\right)-\psi_3\left(\frac{x}{r}\right)\right]\right\}$$

解出$f_1(x)$再代入（5.6）得

$$xf_2(x)=\frac{2\alpha\beta}{\beta-\alpha}\left\{\psi_1\left(\frac{x}{\alpha}\right)-\psi_2\left(\frac{x}{\beta}\right)-\frac{\alpha r}{r-\alpha}\left(\frac{1}{\alpha_2}-\frac{1}{\beta^2}\right)\left[\frac{\alpha\beta}{\beta-\alpha}\left[\psi_1\left(\frac{x}{\alpha}\right)-\psi_2\left(\frac{x}{\beta}\right)\right]\right.\right.$$
$$\left.\left.-\frac{\beta r}{\beta-\alpha}\left[\psi_2\left(\frac{x}{\beta}\right)-\psi_3\left(\frac{x}{r}\right)\right]\right\}\right.$$

$$\frac{1}{8b}\int_0^x f_1(t)dt-\frac{1}{2}\int_0^x f_3(t)dt=\psi_1\left(\frac{x}{\alpha}\right)+\frac{x^2}{4\alpha^2}f_1{}'(x)-\frac{x}{2\alpha}f_2(x)$$

因此可解得$f_1(x)$，再将$f_1(x)$，$f_2(x)$代入（5.2）解出$f_3(x)$，又由于$f_4(x)$可以用$f_1(x)$，$f_2(x)$，$f_3(x)$表之，得

$$u=-\frac{2\alpha r}{3(r-\alpha)}\left(x^3-\frac{y^3}{\alpha^3}\right)\frac{d}{dy}\frac{1}{y^2}\left\{\frac{\alpha\beta}{\beta-\alpha}\left[\psi_1\left(\frac{y}{\alpha}\right)-\psi_2\left(\frac{y}{\beta}\right)\right]-\frac{\beta r}{\beta-r}\times\right.$$
$$\left.\times\left[\psi_2\left(\frac{y}{\beta}\right)-\psi_3\left(\frac{y}{r}\right)\right]\right\}-\frac{\alpha\beta}{\beta-\alpha}\left(x^2-\frac{y^2}{\alpha^2}\right)\frac{d}{dy}\frac{1}{y}\left\{\psi_1\left(\frac{y}{\alpha}\right)-\psi_2\left(\frac{y}{\beta}\right)\right.$$
$$-\frac{\alpha r}{r-\alpha}\left(\frac{1}{\alpha^2}-\frac{1}{\beta^2}\right)\left[\frac{\alpha\beta}{\beta-\alpha}\left[\psi_1\left(\frac{y}{\alpha}\right)-\psi_2\left(\frac{y}{\beta}\right)\right]-\frac{\beta r}{\beta-r}\left[\psi_2\left(\frac{y}{\beta}\right)-\right.\right.$$
$$\left.\left.\left.\psi_3\left(\frac{y}{r}\right)\right]\right\}+2\left(x-\frac{y}{\alpha}\right)\left\{-\frac{\alpha r}{2b(r-\alpha)}\int_0^y\frac{1}{y^2}\left[\frac{\alpha\beta}{\beta-\alpha}\left(\psi_1\left(\frac{y}{\alpha}\right)-\psi_2\left(\frac{y}{\beta}\right)\right)\right.\right.$$
$$\left.-\frac{\beta r}{\beta-r}\left(\psi_2\left(\frac{y}{\beta}\right)-\psi_3\left(\frac{y}{r}\right)\right)dy-\frac{1}{\alpha}\psi_1{}'\left(\frac{y}{\alpha}\right)+\frac{r}{\alpha(r-\alpha)}\times\right.$$
$$\times\frac{d}{dy}\left[\frac{\alpha\beta}{\beta-\alpha}\left(\psi_1\left(\frac{y}{\alpha}\right)-\psi_2\left(\frac{y}{\beta}\right)\right)-\frac{\beta r}{\beta-r}\left(\psi_2\left(\frac{y}{\beta}\right)-\psi_3\left(\frac{y}{r}\right)\right)\right]+$$
$$\frac{\beta}{\beta-\alpha}\frac{d}{dy}\left[\psi_1\left(\frac{y}{\alpha}\right)-\psi_2\left(\frac{y}{\beta}\right)-\frac{\alpha r}{r-\alpha}\left(\frac{1}{\alpha^2}-\frac{1}{\beta^2}\right)\left(\frac{\alpha\beta}{\beta-\alpha}\left(\psi_1\left(\frac{y}{\alpha}\right)-\psi_2\left(\frac{y}{\beta}\right)\right)\right.\right.$$
$$\left.\left.\left.-\frac{\beta r}{\beta-r}\left(\psi_2\left(\frac{y}{\beta}\right)-\psi_3\left(\frac{y}{r}\right)\right)\right)\right]\right\}+\varphi_1\left(\frac{y}{\alpha}\right)$$

$$v=\psi_1\left(\frac{y}{\alpha}\right)-\frac{\alpha r}{r-\alpha}\left(\frac{y^2}{\alpha^2}-x^2\right)\frac{1}{y^2}\left\{\frac{\alpha\beta}{\beta-\alpha}\left(\psi_1\left(\frac{y}{\alpha}\right)-\psi_2\left(\frac{y}{\beta}\right)\right)-\right.$$
$$\left.\frac{\beta r}{\beta-r}\left(\psi_2\left(\frac{y}{\beta}\right)-\psi_3\left(\frac{y}{r}\right)\right)\right\}-\frac{\alpha\beta}{\alpha-\alpha}\left(\frac{y}{\alpha}-x\right)\frac{1}{y}\left\{\psi_1\left(\frac{y}{\alpha}\right)-\psi_2\left(\frac{y}{\beta}\right)\right.$$
$$-\left(\frac{1}{\alpha^2}-\frac{1}{\beta^2}\right)\frac{\alpha r}{\alpha-r}\left[\frac{\alpha\beta}{\beta-\alpha}\left(\psi_1\left(\frac{y}{\alpha}\right)-\psi_2\left(\frac{y}{\beta}\right)\right)-\frac{\beta r}{\beta-r}\left(\psi_2\left(\frac{y}{\beta}\right)-\right.\right.$$
$$\left.\left.\left.\psi_3\left(\frac{y}{r}\right)\right)\right]\right\}$$

附記　若边值条件（5.1）换为

$$\left\{\begin{array}{ll} v|_{y=\alpha x}=\psi_1(x) & v|_{y=\beta x}=\psi_2(x) \\ v|_{y=rx}=\psi_3(x) & u|_{y=\delta x}=\varphi_1(x) \end{array}\right.\quad(-\infty<x<+\infty)\quad(5.8)$$

或者是把（5.1）及（5.8）中的 u 及 v 的位置互調，这样的定解問題仍然是存在唯一的，这儿 φ_1，ψ_i 是三阶可微的連續函数，而且本身直至其二阶导数在零点处为零，α,β,r 是互不相等的非零实数。

§6 第三問題

第三問題是求方程組（Ⅰ）的解 u 及 v，使它們适合于

$$\begin{cases} u\big|_{y=\alpha x} = \varphi_1(x) & u\big|_{y=\beta x} = \varphi_2(x) \\ u\big|_{y=rx} = \varphi_3(x) & u\big|_{y=\delta x} = \varphi_4(x) \end{cases} \quad (-\infty<x<+\infty) \qquad (6.1)$$

这儿 α,β,r 是互不相等的非零实数，φ_i 是三阶可微的函数，而且 $\varphi_i(0)=\varphi_i{}'(0)=\varphi_i{}''(0)=0 \quad (i=1,2,3,4)$

由（Ⅰ）（Ⅲ）及（6.1）得

$$\frac{1}{3!\alpha^3}x^3f_1{}''(x)-\frac{1}{2\alpha^2}x^2f_2{}'(x)+\frac{1}{\alpha}xf_3(x)+f_4(x)=\varphi_1\left(\frac{x}{\alpha}\right) \qquad (6.2)$$

$$\frac{1}{3!\beta^3}x^3f_1{}''(x)-\frac{1}{2\beta^2}x^2f_2{}'(x)+\frac{1}{\beta}xf_3(x)+f_4(x)=\varphi_2\left(\frac{x}{\beta}\right) \qquad (6.3)$$

$$\frac{1}{3!r^3}x^3f_1{}''(x)-\frac{1}{2r^2}x^2f_2{}'(x)+\frac{1}{r}xf_3(x)+f_4(x)=\varphi_3\left(\frac{x}{r}\right) \qquad (6.4)$$

$$\frac{1}{3!\delta^3}x^3f_1{}''(x)-\frac{1}{2\delta^2}x^2f_2{}'(x)+\frac{1}{\delta}xf_3(x)+f_4(x)=\varphi_4\left(\frac{x}{\delta}\right) \qquad (6.5)$$

由（6.2）—（6.5）消去 f_2，f_3，f_4 得

$$f_1{}''(x)=\frac{6r\delta}{\delta-r}\frac{1}{x^3}\left\{\frac{\beta r}{r-\beta}\left[\frac{\alpha\beta}{\beta-\alpha}\left(\varphi_1\left(\left(\frac{x}{\alpha}\right)-\varphi_2\left(\frac{x}{\beta}\right)\right)-\frac{\alpha r}{r-\alpha}\left(\varphi_1\left(\frac{x}{\alpha}\right)-\varphi_2\left(\frac{x}{r}\right)\right)\right]-\frac{\beta\delta}{\delta-\beta}\left[\frac{\alpha\beta}{\beta-\alpha}\left(\varphi_1\left(\frac{x}{\alpha}\right)-\varphi_2\left(\frac{x}{\beta}\right)\right)-\frac{\alpha\delta}{\delta-\alpha}\left(\varphi_1\left(\frac{x}{\alpha}\right)-\varphi_4\left(\frac{x}{\delta}\right)\right)\right]\right\}$$

$$f_2{}'(x)=\frac{1}{3}\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{r}\right)xf_1{}''(x)-\frac{2\beta r}{r-\beta}\frac{1}{x^2}\left\{\frac{\alpha\beta}{\beta-\alpha}\left(\varphi_1\left(\frac{x}{\alpha}\right)-\varphi_2\left(\frac{x}{\beta}\right)\right)-\frac{\alpha r}{r-\alpha}\left(\varphi_1\left(\frac{x}{\alpha}\right)-\varphi_3\left(\frac{x}{r}\right)\right)\right\}$$

$$f_3(x)=\frac{\alpha\beta}{\beta-\alpha}\frac{1}{x}\left\{\varphi_1\left(\frac{x}{\alpha}\right)-\varphi_2\left(\frac{x}{\beta}\right)-\frac{1}{3!}\left(\frac{1}{\alpha^3}-\frac{1}{\beta^3}\right)x^3f_1{}''(x)+\frac{1}{2}\left(\frac{1}{\alpha^2}-\frac{1}{\beta^2}\right)x^2f_2{}'(x)\right\}$$

同时 $f_4(x)$可用 f_1，f_2，f_3 表之

由此可見 u 及 v 都可以用 φ_i 表之，但这一解并不唯一，因为 v 还有一个任意的二次項。

参 考 文 献

〔1〕 华罗庚，吳兹潛，林伟：二阶两个自变数两个未知函数的常系数綫性偏微分方程組的标准型。（未发表）

Системы линейных дифференциальных уравнений второго порядка параболического типа с постоянными коэффициентами

У цзы-цянь　Линь вэй

Резюме

В настоящей статье мы изучим четыре рада краевых задач систем уравнений с частными производными

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\frac{\partial^2}{\partial x^2}\begin{pmatrix}u\\v\end{pmatrix}+2\begin{pmatrix}0 & 1\\b & 0\end{pmatrix}\frac{\partial^2}{\partial x\partial y}\begin{pmatrix}u\\v\end{pmatrix}+\begin{pmatrix}0 & 0\\0 & 4b\end{pmatrix}\frac{\partial^2}{\partial y^2}\begin{pmatrix}u\\v\end{pmatrix}=0 \qquad (1)$$

здесь b —Вещественное, $b\neq 0$. Системы (1)—Канонический вид систем линейных уравнений второго порядка параболического типа с постоянными коэффициентами, характеристическое уравнение которых только имеет один 4-кратный вещественный корень