# 常系数二阶两个自变数两个未知函数的綫性椭圓型微分方程組

## ——特征四次型有一对复重根的情况

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## §1 引 言

假設 A,B,C, 是三个二行二列的实数方陣，則矩陣型式的偏微分方程

$$(\mathrm{I})\qquad A\frac{\partial^2}{\partial x^2}\begin{pmatrix}u\\v\end{pmatrix}+2B\frac{\partial^2}{\partial x\partial y}\begin{pmatrix}u\\v\end{pmatrix}+C\frac{\partial^2}{\partial y^2}\begin{pmatrix}u\\v\end{pmatrix}=0$$

是两个自变数 $x.y$ 两个未知函数 u.v 的两个方程所組成的常系数二阶綫性偏微分方程組。

行列式

$$Q(\xi,\eta)=|A\xi^2+2B\xi\eta+C\eta^2| \tag{1.1}$$

定义为方程組(Ⅰ)的特征四次型

方程組(Ⅰ)与方程組

$$(\mathrm{II})\qquad A_1\frac{\partial^2}{\partial x^2}\begin{pmatrix}u\\v\end{pmatrix}+2B_1\frac{\partial^2}{\partial x\partial y}\begin{pmatrix}u\\v\end{pmatrix}+C_1\frac{\partial^2}{\partial y^2}\begin{pmatrix}u\\v\end{pmatrix}=0$$

的等价关系是由下列三种运算来定义的：i）方程之間的綫性組合：即有一个二行二列的滿秩方陣P左乘方程組(Ⅰ)，使 $A_1=PA$, $B_1=PB$, $C_1=PC$, ii）未知函数的綫性变换：即有一个二行二列的滿秩方陣Q，当作未知函数的綫性变换 $\begin{pmatrix}u\\v\end{pmatrix}=Q\begin{pmatrix}u_1\\v_1\end{pmatrix}$ 时，方程組(Ⅰ)的系数所成的方陣 A,B,C 各变成：$A_1=AQ$，$B_1=BQ$，$C_1=CQ$ iii）自变量的綫性变换：当作自变数的綫性变换 $\begin{pmatrix}x_1\\y_1\end{pmatrix}=\begin{pmatrix}p&q\\r&s\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$，$\begin{vmatrix}p&q\\r&s\end{vmatrix}\neq 0$ 时，方

程組(Ⅰ)的系数所成方陣 A,B,C 各变成：

$A_1 = p^2A+2pqB+q^2C \quad B_1 = prA+(ps+qr)B+qsC, \quad C_1 = r^2A+2rsB+s^2C.$

在文章〔1〕中，已証明若特征四次型有一对复重根时，它的标准型是：

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}\frac{\partial^2}{\partial x^2}\begin{pmatrix}u\\v\end{pmatrix}+2\begin{pmatrix}0&1\\b&0\end{pmatrix}\frac{\partial^2}{\partial x\partial y}\begin{pmatrix}u\\v\end{pmatrix}+\begin{pmatrix}\lambda&0\\0&\mu\end{pmatrix}\frac{\partial^2}{\partial y^2}\begin{pmatrix}u\\v\end{pmatrix}=0 \tag{1.2}$$

$$\lambda+\mu-4b=2, \qquad \lambda\mu=1, \qquad b\neq 0,$$

由于 $4b=\lambda+\frac{1}{\lambda}-2=\frac{(\lambda-1)^2}{\lambda}\neq 0$，故得(1.2)的系数为

$$A=\begin{pmatrix}1&0\\0&1\end{pmatrix} \qquad B=\begin{pmatrix}0&1\\\frac{(\lambda-1)^2}{4\lambda}&0\end{pmatrix} \qquad C=\begin{pmatrix}\lambda&0\\0&\frac{1}{\lambda}\end{pmatrix}, \tag{1.3}$$

对(1.3)左乘以$\begin{pmatrix}\frac{\lambda-1}{2}&0\\0&\lambda\end{pmatrix}$，右乘以$\begin{pmatrix}\frac{2}{\lambda-1}&0\\0&1\end{pmatrix}$，即(1.2)作方程之間的綫性組合及未知函数的綫性变换，立得

$$\begin{pmatrix}1&0\\0&\lambda\end{pmatrix}\frac{\partial^2}{\partial x^2}\begin{pmatrix}u\\v\end{pmatrix}+2\begin{pmatrix}0&\frac{\lambda-1}{2}\\\frac{\lambda-1}{2}&0\end{pmatrix}\frac{\partial^2}{\partial x\partial y}\begin{pmatrix}u\\v\end{pmatrix}+\begin{pmatrix}\lambda&0\\0&1\end{pmatrix}\frac{\partial^2}{\partial y^2}\begin{pmatrix}u\\v\end{pmatrix}=0 \tag{1.4}$$

$$\lambda\neq 0,1$$

在我們的文章[2]中証明：若$\lambda>0$时，(1.4)的 Dirichlet 問題的解是唯一的，本文将証明(1.4)的 Dirichlet 問題的存在性。

我們还做了以下的工作：1)重特征椭园型（特征四次型有一对复重根）方程組的三种边界問題的存在性和唯一性。2)非重特征椭园型方程組（特征四次型有两对复根）的四种边界問題。3)拟椭园型（即 Dirichlet 問題不唯一）方程組的几个边界問題。4)由椭园型方程組所决定的$(\lambda,k)$双解析函数論及其应用。上述各工作以后另文发表。

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## §2　方程組的复数形式及一般解

方程組(1.4)可以写为：

$$\begin{cases}\dfrac{\partial^2 u}{\partial x^2}+(\lambda-1)\dfrac{\partial^2 v}{\partial x\partial y}+\lambda\dfrac{\partial^2 u}{\partial y^2}=0\\[2ex]\lambda\dfrac{\partial^2 v}{\partial x^2}+(\lambda-1)\dfrac{\partial u}{\partial x\partial y}+\dfrac{\partial^2 v}{\partial y^2}=0\end{cases} \tag{2.1}$$

把(2.1)的第二式乘以 i，加到第一式得

$$\frac{\partial^2}{\partial x^2}(u+i\lambda v)+i(\lambda-1)\frac{\partial^2}{\partial x\partial y}(u-iv)+\frac{\partial^2}{\partial y^2}(\lambda u+iv)=0$$

$$\frac{1+\lambda}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u-iv)+\frac{1-\lambda}{4}\left(\frac{\partial^2}{\partial x^2}+2i\frac{\partial^2}{\partial x\partial y}-\frac{\partial^2}{\partial y^2}\right)(u+iv)=0$$

命：

$$u+iv=f(z),\ \frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)=\frac{\partial}{\partial z},\quad \frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)=\frac{\partial}{\partial \bar{z}}。$$

立得

$$(1-\lambda)\frac{\partial^2 f}{\partial \bar{z}^2}+(1+\lambda)\frac{\partial^2 \bar{f}}{\partial \bar{z}\partial z}=0 \tag{2.2}$$

这就是方程組(2.1)的复数形式

由(2.2)出发，可求得方程組的一般解，把(2.2)改写成

$$\frac{\partial}{\partial \bar{z}}\left[(1-\lambda)\frac{\partial f}{\partial \bar{z}}+(1+\lambda)\frac{\partial \bar{f}}{\partial z}\right]=0$$

于是有

$$(1-\lambda)\frac{\partial f}{\partial \bar{z}}+(1+\lambda)\frac{\partial \bar{f}}{\partial z}=\varphi(z) \tag{2.3}$$

这儿 $\varphi(z)$ 是 $z$ 的解析函数，对(2.3)取共軛值得

$$(1+\lambda)\frac{\partial f}{\partial \bar{z}}+(1-\lambda)\frac{\partial \bar{f}}{\partial z}=\overline{\varphi(z)} \tag{2.4}$$

由(2.3)（2.4)立得：

$$\frac{\partial f}{\partial \bar{z}}=\frac{\lambda-1}{4\lambda}\varphi(z)+\frac{\lambda+1}{4\lambda}\overline{\varphi(z)} \tag{2.5}$$

即得

$$f(z)=\frac{\lambda-1}{4\lambda}\bar{z}\varphi(z)+\frac{\lambda+1}{4\lambda}\overline{\Phi(z)}+\psi(z) \tag{2.6}$$

这儿 $\Phi(z)=\int_0^z\varphi(\xi)d\xi$，$\psi(z)$是 $z$ 的解析函数，(2.6)就是方程組(2.1)的一般解。

## §3 Dirichlet 問題

Dirichlet 問題是求函数 $u$、$v$ 使它們在单位园內滿足方程(2.1)及边界条件

$$\left.\begin{aligned} u\big|_{|z|=1} &= h(\zeta) \\ v\big|_{|z|=1} &= g(\zeta) \end{aligned}\right\} \tag{3.1}$$

这儿 $z = x + iy = \rho e^{i\theta}$，$\zeta = e^{it}$，$h(\zeta), g(\zeta)$ 是給定在单位园 $|\zeta| = 1$ 上的函数，其微商 $g'(\zeta), h'(\zeta)$ 滿足 Hölder 条件，即

$$|h'(\zeta_1) - h'(\zeta_2)| \leqslant M_1 \, |\zeta_1 - \zeta_2|^{\mu_1} \quad 0 < \mu_1 \leqslant 1 .$$

$$|g'(\zeta_1) - g'(\zeta_2)| \leqslant M_2 \, |\zeta_1 - \zeta_2|^{\mu_2} \quad 0 < \mu_2 \leqslant 1 .$$

把（2.1）写成

$$\left.\begin{aligned} \Delta u + (\lambda - 1)\left(\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}\right) &= 0 \\ \Delta v + (\lambda - 1)\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2}\right) &= 0 \end{aligned}\right\} \tag{3.2}$$

或

$$\left.\begin{aligned} \lambda \Delta u + (\lambda - 1)\left(\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 u}{\partial x^2}\right) &= 0 \\ \lambda \Delta v + (\lambda - 1)\left(\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 v}{\partial y^2}\right) &= 0 \end{aligned}\right\} \tag{3.3}$$

这里 $\Delta$ 是 Laplace 算符，如果 $u$、$v$ 有三級連續偏微商，由（3.2）及（3.3）有

$$\frac{\partial(\Delta u)}{\partial x} = -(\lambda - 1)\left(\frac{\partial^3 v}{\partial x^2 \partial y} + \frac{\partial^3 u}{\partial x \partial y^2}\right) = \frac{\partial(\Delta v)}{\partial y}$$

$$\frac{\partial(\Delta u)}{\partial y} = -\frac{(\lambda - 1)}{\lambda}\left(\frac{\partial^3 v}{\partial x \partial y^2} - \frac{\partial^3 u}{\partial x^2 \partial y}\right) = -\frac{\partial(\Delta v)}{\partial x}$$

即得

$$\left\{\begin{aligned} \frac{\partial(\Delta u)}{\partial x} &= \frac{\partial(\Delta v)}{\partial y} \\ \frac{\partial(\Delta u)}{\partial y} &= -\frac{\partial(\Delta v)}{\partial x} \end{aligned}\right. \tag{3.4}$$

及

$$\Delta\Delta u = 0 \qquad \Delta\Delta v = 0 .$$

換言之，$\Delta u$ 及 $\Delta v$ 是互为共軛的調和函数，而 $u$、$v$ 是双調和函数。不妨将函数 $u$、$v$ 表成

$$u = (\rho^2 - 1)u_1 + u_2, \qquad v = (\rho^2 - 1)v_1 + v_2, \tag{3.5}$$

这儿 $u_1$、$v_1$、$u_2$、$v_2$ 是单位园內的調和函数，由（3.1）得

$$u_2\big|_{|z|=1} = h(\zeta) \qquad v_2\big|_{|z|=1} = g(\zeta) \tag{3.6}$$

$$\left.\begin{aligned} u_2(z) &= \frac{1}{2\pi}\int_0^{2\pi} h(e^{it})Re\left(\frac{e^{it}+z}{e^{it}-z}\right)dt. \\ v_2(z) &= \frac{1}{2\pi}\int_0^{2\pi} g(e^{it})Re\left(\frac{e^{it}+z}{e^{it}-z}\right)dt. \end{aligned}\right\} \tag{3.7}$$

现在求 $u_1$、$v_1$。首先証明 $u_1$、$v_1$ 是互为共軛的調和函数，由（3.5）有

$$\Delta u = \Delta[(\rho^2-1)u_1] = 4\frac{\partial}{\partial\rho}(\rho u_1)$$

$$\Delta v = \Delta[(\rho^2-1)v_1] = 4\frac{\partial}{\partial\rho}(\rho v_1)$$

由（3.4）利用 Cauchy－Riemann 条件的极坐标形式，有：

$$\rho\frac{\partial}{\partial\rho}\left[4\frac{\partial}{\partial\rho}(\rho u_1)\right] = \frac{\partial}{\partial\theta}\left[4\frac{\partial}{\partial\rho}(\rho v_1)\right]$$

$$\frac{\partial}{\partial\rho}\left[\rho^2\frac{\partial u_1}{\partial\rho}\right] = \frac{\partial}{\partial\rho}\left(\rho\frac{\partial v_1}{\partial\theta}\right)$$

立得：

$$\rho^2\frac{\partial u_1}{\partial\rho} = \rho\frac{\partial v_1}{\partial\theta} + \varphi(\theta)$$

令 $\rho=0$　得　$\varphi(\theta)=0$　于是有

$$\rho^2\frac{\partial u_1}{\partial\rho} = \rho\frac{\partial v_1}{\partial\theta}$$

即

$$\rho\frac{\partial u_1}{\partial\rho} = \frac{\partial v_1}{\partial\theta} \qquad (\rho\neq 0) \tag{3.8}$$

同样，可以求得：

$$\rho\frac{\partial v_1}{\partial\rho} = -\frac{\partial u_1}{\partial\theta} \qquad (\rho\neq 0) \tag{3.9}$$

当 $z\neq 0$ 时 $u_1$、$v_1$ 在 $|z|<1$ 时滿足 Cauchy－Riemann 条件，显然在 $z=0$ 处这一条件也成立，如果用直角坐标形式来表示，即得：

$$\left.\begin{aligned} \frac{\partial u_1}{\partial x} &= \frac{\partial v_1}{\partial y} \\ \frac{\partial u_1}{\partial y} &= -\frac{\partial v_1}{\partial x} \end{aligned}\right\} \quad (|z|<1) \tag{3.10}$$

将（3.5）、（3.7）代入（3.2）的第一式得

$$\Delta[(\rho^2-1)u_1]+(\lambda-1)\left[\frac{\partial^2}{\partial x\partial y}\Big((\rho^2-1)v_1\Big)+\frac{\partial^2}{\partial y^2}\Big((\rho^2-1)u_1\Big)\right]$$

$$+\frac{\lambda-1}{2\pi}\int_0^{2\pi}\left[g(e^{it})\frac{\partial^2}{\partial x\partial y}Re\left(\frac{e^{it}+z}{e^{it}-z}\right)+h(e^{it})\frac{\partial^2}{\partial y^2}Re\left(\frac{e^{it}+z}{e^{it}-z}\right)\right]dt=0$$

利用（3.10）立得

$$4\frac{\partial}{\partial\rho}(\rho u_1)+(\lambda-1)\frac{\partial}{\partial y}(2xv_1+2yu_1)$$

$$=-\frac{\lambda-1}{2\pi}\int_0^{2\pi}\left[g(e^{it})\frac{\partial^2}{\partial x\partial y}Re\left(\frac{e^{it}+z}{e^{it}-z}\right)+h(e^{it})\frac{\partial^2}{\partial y^2}Re\left(\frac{e^{it}+z}{e^{it}-z}\right)\right]dt$$

由

$$\frac{\partial}{\partial y}(xv_1+yu_1)=x\frac{\partial v_1}{\partial y}+y\frac{\partial u_1}{\partial y}+u_1=\rho\frac{\partial u_1}{\partial\rho}+u_1=\frac{\partial}{\partial\rho}(\rho u_1)$$

及恒等式

$$\frac{\partial^2}{\partial y^2}\left(\frac{e^{it}+z}{e^{it}-z}\right)=i\frac{\partial^2}{\partial x\partial y}\left(\frac{e^{it}+z}{e^{it}-z}\right)=(i)^2\frac{\partial^2}{\partial x^2}\left(\frac{e^{it}+z}{e^{it}-z}\right)\tag{3.11}$$

$$=(i)^2\frac{d^2}{dz^2}\left(\frac{e^{it}+z}{e^{it}-z}\right)=\frac{-4e^{it}}{(e^{it}-z)^3}=-2\frac{\partial}{\partial\rho}\left[\rho\frac{2e^{it}-z}{(e^{it}-z)^2}e^{-it}\right]$$

得

$$2(\lambda+1)\frac{\partial}{\partial\rho}(\rho u_1)=-\frac{\lambda-1}{2\pi}\int_0^{2\pi}\Big\{h(e^{it})\frac{\partial}{\partial\rho}Re\left[-2\rho e^{-it}\frac{2e^{it}-z}{(e^{it}-z)^2}\right]+$$

$$+g(e^{it})\frac{\partial}{\partial\rho}Im\left[-2\rho e^{-it}\frac{2e^{it}-z}{(e^{it}-z)^2}\right]\Big\}dt$$

当 $\lambda\neq-1$ 时，

$$\frac{\partial}{\partial\rho}(\rho u_1)=\frac{\lambda-1}{\lambda+1}\frac{1}{2\pi}\int_0^{2\pi}\Big\{h(e^{it})\frac{\partial}{\partial\rho}Re\left[\rho e^{-it}\frac{(2e^{it}-z)}{(e^{it}-z)^2}\right]+$$

$$+g(e^{it})\frac{\partial}{\partial\rho}Im\left[\rho e^{-it}\frac{(2e^{it}-z)}{(e^{it}-z)^2}\right]\Big\}dt$$

两边对 $\rho$ 积分得：

$$\rho u_1=\frac{\lambda-1}{\lambda+1}\frac{1}{2\pi}\int_0^{2\pi}\Big\{h(e^{it})Re\left[\rho e^{-it}\frac{(2e^{it}-z)}{(e^{it}-z)^2}\right]+$$

$$+g(e^{it})Im\left[\rho e^{-it}\frac{(2e^{it}-z)}{(e^{it}-z)^2}\right]\Big\}dt$$

或者写成：

$$u_1=\frac{\lambda-1}{\lambda+1}\frac{1}{2\pi}\int_0^{2\pi}Re\left[\left(h(e^{it})-ig(e^{it})\right)\frac{(2e^{it}-z)}{(e^{it}-z)^2}e^{-it}\right\}dt \quad (3.12)$$

同样将(3.5)、(3.7)代入(3.2)的第二式得：

$$v_1=\frac{\lambda-1}{\lambda+1}\frac{1}{2\pi}\int_0^{2\pi}Im\left[\left(h(e^{it})-ig(e^{it})\right)\frac{(2e^{it}-z)}{(e^{it}-z)^2}e^{-it}\right]dt \quad (3.13)$$

由(3.5)、(3.7)、(3.12)、(3.13)我們得：

$$\begin{cases}u(z)=(\rho^2-1)\dfrac{\lambda-1}{\lambda+1}\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}Re\left[\left(h(e^{it})-ig(e^{it})\right)\frac{(2e^{it}-z)}{(e^{it}-z)^2}e^{-it}\right]dt+\\ \qquad+\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}h(e^{it})R_e\left(\frac{e^{it}+z}{e^{it}-z}\right)dt\\ v(z)=(\rho^2-1)\dfrac{\lambda-1}{\lambda+1}\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}Im\left\{\left[h(e^{it})-ig(e^{it})\right]\left(\frac{2e^{it}-z}{(e^{it}-z)^2}e^{-it}\right)\right\}dt+\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}g(e^{it})Re\left(\frac{e^{it}+z}{e^{it}-z}\right)dt\end{cases} \quad (3.14)$$

这就是 Dirichlet 問題的解。

**定理**，假設 $h(\zeta)$，$g(\zeta)$ 为給定在单位园 $|\zeta|=1$ 上的函数，其导数滿足Hölder条件，則方程組(2.1)

$$\begin{pmatrix}1&0\\0&\lambda\end{pmatrix}\frac{\partial^2}{\partial x^2}\begin{pmatrix}u\\v\end{pmatrix}+2\begin{pmatrix}0&\frac{\lambda-1}{2}\\\frac{\lambda-1}{2}&0\end{pmatrix}\frac{\partial^2}{\partial x\partial y}\begin{pmatrix}u\\v\end{pmatrix}+\begin{pmatrix}\lambda&0\\0&1\end{pmatrix}\frac{\partial^2}{\partial y^2}\begin{pmatrix}u\\v\end{pmatrix}=0$$

$(\lambda\neq0,\ \pm1)$

在单位园 $|z|<1$ 內存在解

$$\left.\begin{aligned}u(z)&=(\rho^2-1)\frac{\lambda-1}{\lambda+1}\frac{1}{2\pi}\int_0^{2\pi}Re\left[\left(h(e^{it})-ig(e^{it})\right)\left(\frac{2e^{it}-z}{(e^{it}-z)^2}e^{-it}\right)\right]dt+\\&\quad+\frac{1}{2\pi}\int_0^{2\pi}u(e^{it})Re\left(\frac{e^{it}+z}{e^{it}-z}\right)dt\\v(z)&=(\rho^2-1)\frac{\lambda-1}{\lambda+1}\frac{1}{2\pi}\int_0^{2\pi}Im\left[\left(h(e^{it})-ig(e^{it})\right)\left(\frac{2e^{it}-z}{(e^{it}-z)^2}\right)e^{-it}\right]dt\\&\quad+\frac{1}{2\pi}\int_0^{2\pi}v(e^{it})Re\left(\frac{e^{it}+z}{e^{it}-z}\right)dt\end{aligned}\right\} \quad (3.14)$$

满足如下的边界条件:

$$u(z)\big|_{|z|=1}=h(\zeta)\ ,\qquad v(z)\big|_{|z|=1}=g(\zeta)$$

証: 命 $f(z)=u(z)+iv(z)$, $f(e^{it})=h(e^{it})+ig(e^{it})$则解(3,14)可写为如下的复数形式:

$$f(z)=(z\bar{z}-1)\frac{\lambda-1}{\lambda+1}\frac{1}{2\pi}\int_0^{2\pi}\overline{f(e^{it})}\frac{2e^{it}-z}{(e^{it}-z)^2}e^{-it}dt+$$

$$+\frac{1}{2\pi}\int_0^{2\pi}f(e^{it})R_e\left(\frac{e^{it}+z}{e^{it}-z}\right)dt\ \text{。}\tag{3.15}$$

1)方程組(2.1)的复数形式为(2.2),由(3.15)得:

$$\frac{\partial^2 f}{\partial^2\bar{z}^2}=\frac{1}{2\pi}\int_0^{2\pi}f(e^{it})\frac{2e^{2it}}{(1-\bar{z}e^{it})^3}dt\tag{3.16}$$

$$\frac{\partial^2 f}{\partial z\partial\bar{z}}=\frac{\lambda-1}{\lambda+1}\frac{1}{2\pi}\int_0^{2\pi}f(e^{it})\frac{2e^{2it}}{(1-\bar{z}e^{it})^3}dt\tag{3.17}$$

将(3.16)、(3.17)代入(2.2)得:

$$(1-\lambda)\frac{\partial^2 f}{\partial\bar{z}^2}+(1+\lambda)\frac{2^2\bar{f}}{\partial z\partial\bar{z}}=0\ \text{。}$$

因此解(3.15)满足方程(2.2),即(3.14)满足(2.1)。

2)由于

$$\frac{1}{2\pi}\int_0^{2\pi}\overline{f(e^{it})}\frac{2e^{it}-z}{(e^{it}-z)^2}e^{-it}dt=\frac{1}{2\pi}\int_0^{2\pi}\overline{f(e^{it})}\left[\frac{1}{(e^{it}-z)^2}+\frac{e^{-it}}{(e^{it}-z)}\right]dt$$

$$=\frac{-1}{2\pi i}\frac{(\overline{f(e^{it})e^{it}})}{e^{it}-z}\Bigg|_0^{2\pi}+\frac{1}{2\pi i}\int_0^{2\pi}\left(\frac{d}{dt}\overline{f(e^{it})e^{it}}\right)\frac{dt}{(e^{it}-z)}+$$

$$+\frac{1}{2\pi}\int_0^{2\pi}\frac{\overline{f(e^{it})e^{it}}}{e^{it}-z}dt$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\frac{\overline{-f'(e^{it})e^{i2t}}-\overline{f(e^{it})e^{it}}}{e^{it}-z}dt+\frac{1}{2\pi}\int_0^{2\pi}\frac{\overline{f(e^{it})e^{it}}}{e^{it}-z}dt$$

$$=\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{\overline{-f'(\zeta)\zeta^3}}{(\zeta-z)}d\zeta$$

及 $f'(\zeta)$满足 Hölder 条件,因此按照 Сохоцкий 公式:

$$\lim_{z\to e^{i\theta}}\frac{1}{2\pi}\int_0^{2\pi}\overline{f(e^{it})}\,\frac{2e^{it}-z}{(e^{it}-z)^2}\,e^{-it}dt=\lim_{z\to e^{i\theta}}\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{\overline{-f'(\zeta)\zeta^3}}{(\zeta-z)}\,d\zeta$$

有界，由(3.15)得：

$$f(z)\big|_{|z|=1}=\lim_{\rho\to1}f(z)=$$

$$\lim_{\rho\to1}(\rho^2-1)\frac{\lambda-1}{\lambda+1}\frac{1}{2\pi}\cdot$$

$$\cdot\int_0^{2\pi}\overline{f(e^{it})}\left(\frac{2e^{it}-z}{(e^{it}-z)^2}\,e^{-it}dt\right)+$$

$$+\lim_{\rho\to1}\frac{1}{2\pi}\int_0^{2\pi}f(e^{it})R_e\left(\frac{e^{it}+z}{e^{it}-z}\right)dt$$

$$=f(\zeta)=h(\zeta)+ig(\zeta)\text{。}$$

因此(3.14)滿足边界(3.1)，明所欲証。

## §4 Fourier 方法

Dirichlet 問題的解(3.14)可用 Fourier 方法获得，这就是从方程組(2.1)的一般解(2.6)

$$f(z)=\frac{\lambda-1}{4\lambda}\bar z\varphi(z)+\frac{\lambda+1}{4\lambda}\overline{\int_0^z\varphi(\xi)d\xi}+\psi(z)$$

出发，命

$$\varphi(z)=\sum_{n=0}^{\infty}a_nz^n\qquad\psi(z)=\sum_{n=0}^{\infty}b_nz^n\qquad(4.1)$$

即得：

$$f(z)=\frac{\lambda-1}{4\lambda}\bar z\sum_{n=0}^{\infty}a_nz^n+\frac{\lambda+1}{4\lambda}\sum_{n=0}^{\infty}\frac{\bar a_n}{n+1}\bar z^{n+1}+\sum_{n=0}^{\infty}b_nz^n$$

$$=\sum_{n=0}^{\infty}\left(\frac{\lambda-1}{4\lambda}a_{n+1}\rho^{n+2}+b_n\rho^n\right)e^{in\theta}+$$

$$+\left(\frac{\lambda+1}{4\lambda}\bar a_0\rho+\frac{\lambda-1}{4\lambda}a_0\rho\right)e^{-i\theta}+\sum_{n=2}^{\infty}\frac{\lambda+1}{4\lambda}\frac{\bar a_{n-1}}{n}\rho^ne^{-in\theta}$$

命 $f(\zeta)=h(e^{it})+ig(e^{it})$ (3.1)得：

$$f(\zeta)=f(z)\Big|_{|z|=1}=\sum_{n=0}^{\infty}\left(\frac{\lambda-1}{4\lambda}a_{n+1}+b_n\right)e^{in\theta}+$$

$$+\left(\frac{\lambda+1}{4\lambda}\overline{a_0}+\frac{\lambda-1}{4\lambda}a_0\right)e^{-i\theta}+\sum_{n=2}^{\infty}\frac{\lambda+1}{4\lambda}\frac{\overline{a_{n-1}}}{n}e^{-in\theta}$$

立得：

$$\frac{\lambda+1}{4\lambda}\frac{\overline{a}_{n-1}}{n}=\frac{1}{2\pi}\int_0^{2\pi}f(e^{it})e^{int}dt,\ n\geqslant 2. \tag{4.2}$$

$$\frac{\lambda+1}{4\lambda}\overline{a_0}+\frac{\lambda-1}{4\lambda}a_0=\frac{1}{2\pi}\int_0^{2\pi}f(e^{it})e^{it}dt \tag{4.3}$$

$$\frac{\lambda-1}{4\lambda}a_{n+1}+b_n=\frac{1}{2\pi}\int_0^{2\pi}f(e^{it})e^{-int}dt\quad n\geqslant 0. \tag{4.4}$$

当$\lambda\neq-1$时，即得：

$$a_n=\frac{1}{2\pi}\int_0^{2\pi}\frac{4\lambda}{\lambda+1}\overline{f(e^{it})}\,(n+1)e^{-i(n+1)t}\,dt,n\geqslant 1 \tag{4.5}$$

$$b_n=\frac{1}{2\pi}\int_0^{2\pi}f(e^{it})e^{-int}\,dt- \tag{4.6}$$

$$-\frac{1}{2\pi}\int_0^{2\pi}\frac{\lambda-1}{\lambda+1}(n+2)\overline{f(e^{it})}\,e^{-i(n+2)t}\,dt\quad n\geqslant 0$$

代入(4.1)得：

$$\varphi(z)=a_0+\sum_{n=1}^{\infty}\frac{1}{2\pi}\int_0^{2\pi}\frac{4\lambda}{\lambda+1}\overline{f(e^{it})}\,(n+1)e^{-i(n+1)t}z^n dt$$

$$=a_0+\frac{1}{2\pi}\int_0^{2\pi}\frac{4\lambda}{\lambda+1}\overline{f(e^{it})}\,\frac{1}{z}\sum_{n=1}^{\infty}(n+1)(ze^{-it})^{n+1}dt$$

$$\varphi(z)=a_0+\frac{1}{2\pi}\int_0^{2\pi}\frac{4\lambda}{\lambda+1}\overline{f(e^{it})}\,e^{-i2t}\frac{2z-z^2e^{-it}}{(1-ze^{-it})^2}dt \tag{4.7}$$

$$\int_0^z\varphi(\xi)d\xi=a_0z+\frac{1}{2\pi}\int_0^{2\pi}\frac{4\lambda}{\lambda+1}\overline{f(e^{it})}\,\frac{(ze^{-it})^2}{1-ze^{-it}}dt \tag{4.8}$$

而

$$\psi(z)=\frac{1}{2\pi}\int_0^{2\pi}f(e^{it})\sum_{n=0}^{\infty}(ze^{-it})^n dt-$$

$$-\frac{1}{z^2}\frac{1}{2\pi}\int_0^{2\pi}\frac{\lambda-1}{\lambda+1}\overline{f(e^{it})}\sum_{n=0}^{\infty}(n+2)(ze^{-it})^{n+2}dt$$

$$\psi(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\frac{dn}{1-ze^{-it}} \tag{4.9}$$

$$-\frac{1}{2\pi}\int_0^{2\pi}\frac{\lambda-1}{\lambda+1}\overline{f(e^{it})}\, e^{-i2t}\frac{2-ze^{-it}}{(1-ze^{-it})^2}\,dt$$

将(4.7)、(4.8)、(4.9)代入(2.6)得:

$$f(z) = \frac{\lambda-1}{4\lambda} a_0\bar{z} + \frac{\lambda-1}{\lambda+1}\frac{1}{2\pi}\int_0^{2\pi}\overline{f(e^{it})}\, e^{-i2t}\frac{2-ze^{-it}}{(1-ze^{-it})^2} z\bar{z}\,dt +$$

$$+\frac{\lambda+1}{4\lambda}\bar{a}_0\bar{z} + \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\left[\overline{\frac{(ze^{-it})^2}{1-ze^{-it}}}\right]dt +$$

$$+\frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\frac{dt}{1-ze^{-it}} - \frac{1}{2\pi}\int_0^{2\pi}\frac{\lambda-1}{\lambda+1}\overline{f(e^{it})}\, e^{-i2t}\frac{2-ze^{-it}}{(1-ze^{-it})^2}\,dt$$

由(4.3)得

$$f(z) = (z\bar{z}-1)\left(\frac{\lambda-1}{\lambda+1}\right)\frac{1}{2\pi}\int_0^{2\pi}\overline{f(e^{it})}\, e^{-i2t}\frac{2-ze^{-it}}{(1-ze^{-it})^2}\,dt +$$

$$+\frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\,(\overline{ze^{-it}})\,dt + \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\left[\frac{1}{1-ze^{-it}} + \left(\overline{\frac{(ze^{-it})^2}{1-ze^{-it}}}\right)\right]dt$$

即得:

$$f(z) = (z\bar{z}-1)\left(\frac{\lambda-1}{\lambda+1}\right)\frac{1}{2\pi}\int_0^{2\pi}\overline{f(e^{it})}\,\frac{2e^{it}-z}{(e^{it}-z)^2}\,e^{it}dt + \tag{4.10}$$

$$+\frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\,Re\left(\frac{e^{it}+z}{e^{it}-z}\right)dt$$

这正好是解的复数形式(3.15)

附記．利用$\lambda$,k型双解析函数来求解的問題另文发表。

## 参 考 文 献

〔1〕 华罗庚　吳茲潛　林　伟　二阶两个自变数两个未知函数常系数的綫性偏微分方程組的标准型(待发表)

〔2〕 华罗庚　吳茲潛　林　伟　常系数二阶椭园型偏微分方程組的Dirichlet問題的唯一性定理

# Системы линейных дифференциальных уравнений второго порядка эллиптического типа с постоянными коэффициентами

Линь Вэй  у цзы-цянь


Резюме

В настоящей статье мы докажем существование решения задачи Дирихле систем уравнений с частными производными

$$(1)\quad \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \frac{\partial^2}{\partial x^2} \begin{pmatrix} u \\ v \end{pmatrix} + 2 \begin{pmatrix} 0 & \frac{\lambda-1}{2} \\ \frac{\lambda-1}{2} & 0 \end{pmatrix} \frac{\partial^2}{\partial x \partial y} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial^2}{\partial y^2} \begin{pmatrix} u \\ v \end{pmatrix} = 0$$

(здесь $\lambda \neq 0, \pm 1$), причём зададим его представлений. Системы (1)—Канонический вид систем линейных уравнений второго порядка эллиптического типа с постоянными коэффициентами, характеристическое уравнение которых имеет два кратных комплексных корня.