网架结构分析的拟夹层板法

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平板型网架结构是高次超静定的空间杆系结构。分析这种结构的力学模型基本上有两种：一种是离散化模型，把网架作为空间桁架进行分析；另一种是连续化模型，将网架近似地视为一连续体进行分析。目前国内外常用的连续化方法大致有两种：一种是将网架化成交叉梁、各向同性板或各向异性板，通过网架结构标准单元与连续体单元间所建立的力与变形的关系，来求得换算的弹性模量、泊桑比及板的折算厚度等；另一种是从离散化模型出发，首先建立网架的差分方程式，然后转换为近似的微分方程式。这两种方法各有优缺点，但都比较麻烦，而且不能以一般而统一的格式来建立各种网架的微分方程式。

本文是将网架视作夹层板，通过网格连续化的一般方法求出有关刚度系数，然后利用夹层板理论，以一般而统一的格式来建立各种网架的微分方程式。

根据网架结构的受力特性，可将网架的上、下弦面看成夹层板的上、下表层，而将中间腹杆层看成夹心，并假定：

1．表层只有抗拉压刚度，且应力沿表层均匀分布，即假定表层处于薄膜应力状态。

2．夹心只能抗剪。

一、网格的连续化

在建立网架的夹层板模型时，应先将上、下弦网格连续化为弹性薄膜，薄膜的弹性矩阵D可按下式计算

$$\mathbf{D}=\mathbf{C}^T\mathbf{K}\mathbf{C} \qquad (1.1)$$

式中

$$\mathbf{D}=\begin{pmatrix} D_{11} & D_{12} & D_{13} \\ 对 & D_{22} & D_{23} \\ & 称 & D_{33} \end{pmatrix} \qquad \mathbf{C}=\begin{pmatrix} C^2_{12x} & C^2_{12y} & C_{12x}G_{12y} \\ C^2_{23x} & C^2_{23y} & C_{23x}G_{23y} \\ \vdots & \vdots & \vdots \\ C^2_{ijx} & C^2_{ijy} & C_{ijx}G_{ijy} \end{pmatrix}_{n\times 3}$$

$$K=\begin{pmatrix} \frac{EF_{12}}{a_{12}} & & & \bigcirc \\ & \frac{EF_{23}}{a_{23}} & & \\ & & \ddots & \\ \bigcirc & & & \frac{EF_{ij}}{a_{ij}} \end{pmatrix}_{n\times n}$$

其中 E 为杆件弹性模量，F_{ij}为ij杆的截面积，a_{ij}为ij方向杆件的间距，C_{ijx}及C_{ijy}是ij杆的方向余弦。

为了便于应用，现将各种常用网格的有关刚度系数列于下：

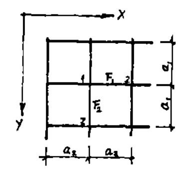


图　1

1．两向正交正放网格（图1）

$$D_{11}=\frac{EF_1}{a_1},\quad D_{22}=\frac{EF_2}{a_2},\quad D_{12}=D_{13}=D_{23}=D_{33}=0 \tag{1.2}$$

2．两向正交斜放网格（图2）

$$D_{11}=D_{22}=D_{33}=D_{12}=\frac{E}{4}\left(\frac{F_1}{a_1}+\frac{F_2}{a_2}\right),\quad D_{13}=D_{23}=\frac{E}{4}\left(\frac{F_2}{a_2}-\frac{F_1}{a_1}\right) \tag{1.3}$$

当　$F_1=F_2=F, a_1=a_2=a$时，$D_{11}=D_{22}=D_{33}=D_{12}=\frac{EF}{2a}$，$D_{13}=D_{23}=0$　(1.4)

3．两向斜交斜放网格（图3）

$$D_{11}=d_1,\quad D_{12}=D_{33}=\frac{d_1}{3},\quad D_{22}=\frac{d_1}{9},\quad D_{13}=3d_2,\quad D_{23}=d_2 \tag{1.5}$$

式中　$d_1=\frac{9E}{16}\left(\frac{F_1}{a_1}+\frac{F_2}{a_2}\right)$，$d_2=\frac{\sqrt{3}\,E}{16}\left(\frac{F_2}{a_2}-\frac{F_1}{a_1}\right)$

当　$F_1=F_2=F$，$a_1=a_2=a$时，$d_1=\frac{9EF}{8a}$，$d_2=0$　(1.6)

4．正三角形网格（图4）、**正三角形——六角形网格**（图5）

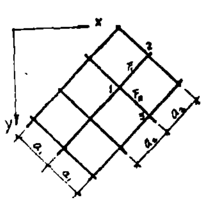


图　2

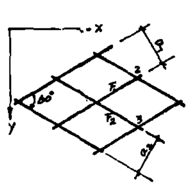


图　3

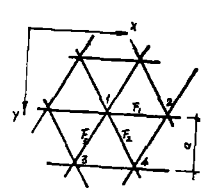


图　4

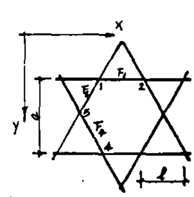


图　5

$$D_{11}=\frac{EF_1}{a}+d_1,\ D_{22}=9d_1,\ D_{12}=D_{33}=3d_1,\ D_{13}=d_2,\ D_{23}=3d_2 \qquad (1.7)$$

式中 $$d_1=\frac{E}{16}\left(\frac{F_2+F_3}{a}\right),\ d_2=\frac{\sqrt{3}E}{16a}(F_2-F_3)$$

当 $$F_1=F_2=F_3=F\text{时},\ D_{11}=D_{22}=\frac{9EF}{8a},\ D_{12}=D_{33}=\frac{3EF}{8a},\ D_{13}=D_{23}=0 \qquad (1.8)$$

二、不考虑剪切变形时网架微分方程式的建立

1. 网架一般微分方程式的建立

由弹性力学可知，上表层的内力与应变的关系为

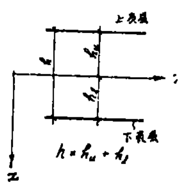


图 6

$$\begin{Bmatrix} T^u_{xx} \\ T^u_{yy} \\ T^u_{xy} \end{Bmatrix} = \begin{bmatrix} D^u_{11} & D^u_{22} & D^u_{13} \\ \text{对} & D^u_{22} & D^u_{32} \\ & \text{称} & D^u_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon^u_x \\ \varepsilon^u_y \\ \varepsilon^u_{xy} \end{Bmatrix} \qquad (2.1.1)$$

下表层的内力与应变的关系为

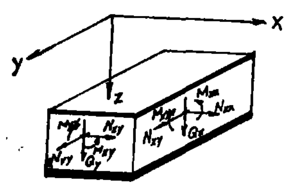


图 7

$$\begin{Bmatrix} T^l_{xx} \\ T^l_{yy} \\ T^l_{xy} \end{Bmatrix} = \begin{bmatrix} D^l_{11} & D^l_{12} & D^l_{13} \\ \text{对} & D^l_{22} & D^l_{23} \\ & \text{称} & D^l_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon^l_x \\ \varepsilon^l_y \\ \varepsilon^l_{xy} \end{Bmatrix} \qquad (2.1.2)$$

设u、v、w为xy平面内任一点沿x,y,z轴方向的位移，则上表层的应变为

$$\varepsilon^u_x=\frac{\partial u}{\partial x}+h_u\frac{\partial^2 w}{\partial x^2},\quad \varepsilon^u_y=\frac{\partial v}{\partial y}+h_u\frac{\partial^2 w}{\partial y^2},\quad \varepsilon^u_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+2h_u\frac{\partial^2 w}{\partial x\partial y} \qquad (2.1.3)$$

而下表层的应变为

$$\varepsilon^l_x=\frac{\partial u}{\partial x}-h_l\frac{\partial^2 w}{\partial x^2},\quad \varepsilon^l_y=\frac{\partial v}{\partial y}-h_l\frac{\partial^2 w}{\partial y^2},\quad \varepsilon^l_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}-2h_l\frac{\partial^2 w}{\partial x\partial y} \qquad (2.1.4)$$

根据夹层板理论，板的断面内力为

$$N_{xx}=T^u_{xx}+T^l_{xx},\ N_{yy}=T^u_{yy}+T^l_{yy},\ N_{xy}=N^u_{xy}+N^l_{xy}$$

$$M_{xx}=T^l_{xx}h_l-T^u_{xx}h_u,\ M_{yy}=T^l_{yy}h_l-T^u_{yy}h_u,\ M_{xy}=T^l_{xy}h_l-T^u_{xy}h_u \qquad (2.1.5)$$

将(2.1.1)，(2.1.2)，(2.1.3)，(2.1.4)式代入(2.1.5)式，可得下式

$$\begin{Bmatrix} \mathbf{N} \\ \mathbf{M} \end{Bmatrix} = \begin{bmatrix} \mathbf{P} & \mathbf{S} \\ \mathbf{S} & \mathbf{B} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\chi} \end{Bmatrix} \tag{2.1.6}$$

式中

$$\mathbf{N} = \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} \quad \mathbf{M} = \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} \quad \boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \quad \boldsymbol{\chi} = \begin{Bmatrix} -\dfrac{\partial^2 w}{\partial x^2} \\ -\dfrac{\partial^2 w}{\partial y^2} \\ -2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \tag{2.1.7}$$

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ 对 & p_{22} & p_{23} \\ & 称 & p_{33} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 对 & b_{22} & b_{23} \\ & 称 & b_{33} \end{bmatrix} \quad \mathbf{S} = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ 对 & s_{22} & s_{23} \\ & 称 & s_{33} \end{bmatrix} \tag{2.1.8}$$

其中$p_{ij} = D^u_{ij} + D^l_{ij}$，$b_{ij} = D^u_{ij}\, h^2_u + D^l_{ij}\, h^2_l$，$S_{ij} = D^l_{ij}\, h_l - D^u_{ij} h_u$

由(2.1.6)式可知，当矩阵S的元素全为零时，板弯曲作用与膜作用就可分离，即

$$\frac{h_u}{h_l} = \frac{D^l_{11}}{D^u_{11}} = \frac{D^l_{12}}{D^u_{12}} = \frac{D^l_{13}}{D^u_{13}} = \frac{D^l_{22}}{D^u_{22}} = \frac{D^l_{23}}{D^u_{23}} = \frac{D^l_{33}}{D^u_{33}} \tag{2.1.9}$$

而板的平衡微分方程为

$$\left.\begin{aligned} &\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} + X = 0\,, \quad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} + Y = 0 \\ &\frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + q = 0 \end{aligned}\right\} \tag{2.1.10}$$

将(2.1.6)式代入(2.1.10)式，经整理后可得

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 对 & a_{22} & a_{23} \\ & 称 & a_{33} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} -x \\ -y \\ q \end{Bmatrix} \tag{2.1.11}$$

式中$a_{11} = p_{11}\dfrac{\partial^2}{\partial x^2} + 2p_{13}\dfrac{\partial^2}{\partial x \partial y} + p_{33}\dfrac{\partial^2}{\partial y^2}$，

$$a_{12} = p_{13}\frac{\partial^2}{\partial x^2} + (p_{12} + p_{33})\frac{\partial^2}{\partial x \partial y} + p_{23}\frac{\partial^2}{\partial y^2}$$

$$a_{13} = -s_{11}\frac{\partial^3}{\partial x^3} - 3s_{13}\frac{\partial^3}{\partial x^2 \partial y} - (2s_{33} + s_{12})\frac{\partial^3}{\partial x \partial y^2} - s_{23}\frac{\partial^3}{\partial y^3}\,,$$

$$a_{22} = p_{33}\frac{\partial^2}{\partial x^2} + 2p_{23}\frac{\partial^2}{\partial x \partial y} + p_{22}\frac{\partial^2}{\partial y^2}$$

$$a_{23} = -s_{13}\frac{\partial^3}{\partial x^3} - (2s_{33} + s_{12})\frac{\partial^3}{\partial x^2 \partial y} - 3s_{23}\frac{\partial^3}{\partial x \partial y^2} - s_{22}\frac{\partial^3}{\partial y^3}$$

$$a_{33}=b_{11}\frac{\partial^4}{\partial x^4}+4b_{13}\frac{\partial^4}{\partial x^3\partial y}+2(b_{12}+2b_{33})\frac{\partial^4}{\partial x^2\partial y^2}+4b_{23}\frac{\partial^4}{\partial x\partial y^3}+b_{22}\frac{\partial^4}{\partial y^4} \quad (2.1.12)$$

(2.1.11)式就是以u、v、w为基本未知量的网架一般微分方程式。若将各种网架的$\mathbf{D}^u$, $\mathbf{D}^l$代入上式，就能得到各种网架的微分方程式。

从(2.1.11)可知，位移u、v和w不耦合的条件，就是a_{13}、a_{23}中的系数全为零，即

$$\frac{h_u}{h_l}=\frac{D^l_{11}}{D^u_{11}}=\frac{D^l_{22}}{D^u_{22}}=\frac{D^l_{13}}{D^u_{13}}=\frac{D^l_{23}}{D^u_{23}}=\frac{2D^l_{33}+D^l_{12}}{2D^u_{33}+D^u_{12}} \quad (2.1.13)$$

顺便指出，(2.1.9)式要比(2.1.13)式要求严些，满足(2.1.9)式就一定能满足(2.1.13)式。

2. 各种网架的微分方程式

为了便于应用，下面给出各种网架的微分方程式。

(1)两向正交正放网架(图8)、正放四角锥网架(图9)、正放抽空四角锥网架(图10)

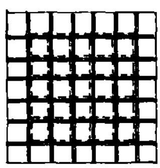


图8 两向正交正放网架

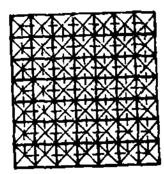


图9 正放四角锥网架

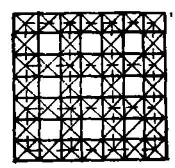


图10 正放抽空四角锥网架

当$E^u=E^l=E$和满足u、v与w不耦合的条件时，有

$$A_1\frac{\partial^2 u}{\partial x^2}+X=0,\quad A_2\frac{\partial^2 v}{\partial y^2}+Y=0 \quad (2.2.1)$$

$$B_1\frac{\partial^4 w}{\partial x^4}+B_2\frac{\partial^4 w}{\partial y^4}=q \quad (2.2.2)$$

对于两向正交正放网架、正放四角锥网架，方程式中的系数为

$$\left.\begin{aligned}A_1&=\frac{E}{a_1}(F^u_1+F^l_1),\quad B_1=\frac{E}{a_1}(F^u_1h_u^2+F^l_1h_l^2)\\A_2&=\frac{E}{a_2}(F^u_2+F^l_2),\quad B_2=\frac{E}{a_2}(F^u_2h_u^2+F^l_2h_l^2)\end{aligned}\right\} \quad (2.2.3)$$

当 $F^u_1=F^u_2=F^u$，$F^l_1=F^l_2=F^l$，$a_1=a_2=a$时

$$\frac{h_u}{h_l}=\frac{F^l}{F^u}=n,\quad h_u=\frac{nh}{1+n},\quad h_l=\frac{h}{1+n} \quad (2.2.4)$$

$$J=F^uh_u^2+F^lh_l^2=F^u\frac{nh^2}{1+n}=F^l\frac{h^2}{1+n} \quad (2.2.5)$$

$$A_1=A_2=\frac{E}{a}(F^u+F^l)=A,\quad B_1=B_2=\frac{EJ}{a}=B \quad (2.2.6)$$

对于正放抽空四角锥网架，方程式中的系数为

$$\left.\begin{aligned}A_1=\frac{E}{a_1}\left(F_1^u+\frac{F_1^l}{2}\right),\quad B_1=\frac{E}{a_1}\left(F_1^u h_u^2+\frac{F_1^l h_l^2}{2}\right)\\ A_2=\frac{E}{a_2}\left(F_2^u+\frac{F_2^l}{2}\right),\quad B_2=\frac{E}{a_2}\left(F_2^u h_u^2+\frac{F_2^l h_l^2}{2}\right)\end{aligned}\right\}\quad(2.2.7)$$

而当 $F_1^u=F_2^u=F^u$，$F_1^l=F_2^l=F^l$，$a_1=a_2=a$ 时，有

$$\frac{h_u}{h_l}=\frac{F^l}{2F^u}=\frac{n}{2},\quad h_u=\frac{nh}{2+n},\quad h_l=\frac{2h}{2+n}\qquad(2.2.8)$$

$$J=F^u h_u^2+\frac{F^l h_l^2}{2}=F^l\frac{h^2}{2+n}=F^u\frac{nh^2}{2+n}\qquad(2.2.9)$$

$$A_1=A_2=\frac{E}{a}\left(F^u+\frac{F^l}{2}\right)=A,\quad B_1=B_2=\frac{EJ}{a}=B\qquad(2.2.10)$$

若网架没有水平力作用，即 $x=y=0$时，只需求解方程(2.2.2)。

(2) 两向正交斜放网架（图11）

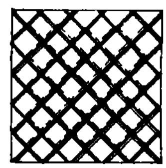


图11　两向正交斜放网架

当$E^u=E^l=E$，$F_1^u=F_2^u=F^u$，$F_1^l=F_2^l=F^l$时，有

$$\frac{A}{2}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+A\frac{\partial^2 v}{\partial x\partial y}+X=0,$$

$$A\frac{\partial^2 u}{\partial x\partial y}+\frac{A}{2}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+Y=0\qquad(2.2.11)$$

$$\frac{\partial^4 w}{\partial x^4}+6\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{q}{B}\qquad(2.2.12)$$

式中的系数

$$A=\frac{E}{a}(F^u+F^l),\quad B=\frac{EJ}{2a}\ (J\text{按}(2.2.5)\text{式计算})\qquad(2.2.13)$$

方程(2.2.12)与通常按交叉梁通过转轴公式所导出的相同。没有水平力作用，即$x=y=0$时，只需求解方程(2.2.12)。

(3) 两向斜交斜放网架（图12）

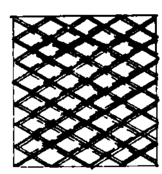


图12　两向斜交斜放网架

当$\alpha=60°$，$E^u=E^l=E$，$F_1^u=F_2^u=F^u$，$F_1^l=F_2^l=F^l$时，有

$$A\left(\frac{\partial^2 u}{\partial x^2}+\frac{1}{3}\frac{\partial^2 u}{\partial y^2}\right)+\frac{2A}{3}\frac{\partial^2 v}{\partial x\partial y}+X=0,$$

$$\frac{2A}{3}\frac{\partial^2 u}{\partial x\partial y}+A\left(\frac{1}{3}\frac{\partial^2 v}{\partial x^2}+\frac{1}{9}\frac{\partial^2 v}{\partial y^2}\right)+Y=0\qquad(2.2.14)$$

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{1}{9}\frac{\partial^4 w}{\partial y^4}=\frac{q}{B} \qquad (2.2.15)$$

式中的系数

$$A=\frac{8E}{9a}(F^u+F^l),\ B=\frac{9EJ}{8a}(J\text{按(2.2.5)式计算}) \qquad (2.2.16)$$

(4)三向网架(图13)、三角锥网架(图14)、抽空三角锥网架(图15)

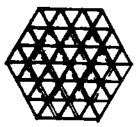


图13 三向网架

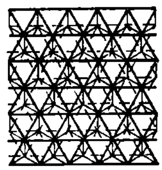


图14 三角锥网架

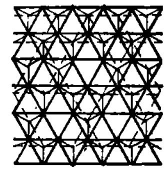


图15 抽空三角锥网架

$$A\left(\frac{\partial^2 u}{\partial x^2}+\frac{1}{3}\frac{\partial^2 u}{\partial y^2}\right)+\frac{2A}{3}\frac{\partial^2 v}{\partial x\partial y}+X=0,$$

$$\frac{2A}{3}\frac{\partial^2 u}{\partial x\partial y}+A\left(\frac{1}{3}\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+Y=0 \qquad (2.2.17)$$

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}=q/B \qquad (2.2.18)$$

对于三向网架、三角锥网架，式中系数为

$$A=\frac{3\sqrt{3}E(F^u+F^l)}{4l},\ B=\frac{3\sqrt{3}}{4}\frac{nEF^u h^2}{(1+n)l} \qquad (2.2.19)$$

对于抽空三角锥网架，式中系数为

$$A=\frac{3\sqrt{3}}{4}\frac{E\left(F^u+\frac{F^l}{2}\right)}{l},\ B=\frac{3\sqrt{3}}{4}\frac{nEF^u h^2}{(2+n)l} \qquad (2.2.20)$$

其中l为上弦网格的边长。

当没有水平力作用时，只需求解方程(2.2.18)。

(5)斜放四角锥网架(图16)及星形四角锥架网(图17)

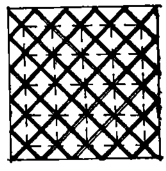


图16 斜放四角錐网架

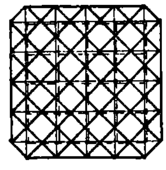


图17 星形四角錐网架

当$E^u=E^l=E$, $F_1^u=F_2^u=F^u$, $F_1^l=F_2^l=F^l$, $\frac{h_u}{h_l}=\frac{\sqrt{2}F^l}{F^u}$时，有

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \text{对} & a_{22} & a_{23} \\ & \text{称} & a_{33} \end{bmatrix}\begin{Bmatrix} u \\ v \\ w \end{Bmatrix}=\begin{Bmatrix} -x \\ -y \\ q \end{Bmatrix} \qquad (2.2.21)$$

式中 $a_{11}=d^u(1+\sqrt{2}n)\frac{\partial^2}{\partial x^2}+d^u\frac{\partial^2}{\partial y^2}$，$a_{12}=2d^u\frac{\partial^2}{\partial x\partial y}$，$a_{13}=3d^uhu\frac{\partial^4}{\partial x\partial y^2}$，

$$a_{22}=d^u\frac{\partial^2}{\partial x^2}+d^u(1+\sqrt{2}n)\frac{\partial^2}{\partial y^2},\quad a_{23}=3d^uhu\frac{\partial^3}{\partial x^2\partial y},$$

$$a_{33}=\frac{2nd^uh^2}{\sqrt{2}(1+\sqrt{2}n)}\frac{\partial^4}{\partial x^4}+6d^uhu^2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{2nd^uh^2}{\sqrt{2}(1+\sqrt{2}n)}\frac{\partial^4}{\partial y^4} \qquad (2.2.22)$$

其中 $d^u=\frac{EF^u}{2a}$，$h_u=\frac{\sqrt{2}nh}{1+\sqrt{2}n}$

这种网架的u、v与w是耦合的，因a_{13}、a_{23}中的系数不可能为零。

若$x=y=0$，并设

$$u=\frac{3hu}{1+\sqrt{2}u}\frac{\partial^5\phi}{\partial x^3\partial y^2}-3hu\frac{\partial^5\phi}{\partial x\partial y^4},\quad v=\frac{3hu}{1+\sqrt{2}n}\frac{\partial^5\phi}{\partial x^2\partial y^3}-3hu\frac{\partial^5\phi}{\partial x^4\partial y}$$

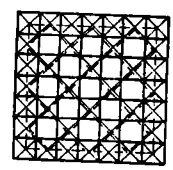
$$w=\frac{\partial^4\phi}{\partial x^4}+\left[(1+\sqrt{2}n)-\frac{3}{(1+\sqrt{2}n)}\right]\frac{\partial^4\phi}{\partial x^2\partial y^2}+\frac{\partial^4\phi}{\partial y^4} \qquad (2.2.23)$$

将（2.2.23）式代入（2.2.21）式，可使（2.2.21）式中的前两式变为恒等式，而第三个方程变为

$$\frac{\partial^8\phi}{\partial x^8}+\frac{\partial^8\phi}{\partial y^8}+2(1+3\sqrt{2n})\frac{\partial^8\phi}{\partial x^4\partial y^4}+(\sqrt{2}n-2)\left(\frac{\partial^8\phi}{\partial x^6\partial y^2}+\frac{\partial^8\phi}{\partial x^2\partial y^6}\right)=\frac{q}{B} \qquad (2.2.24)$$

其中 $B=\frac{EF^u}{a}\frac{nh^2}{\sqrt{2}(1+\sqrt{2}n)}$

（6）棋盘形四角锥网架（图18）



当$E^u=E^l=E$，$F_1^u=F_2^u=F^u$，$F_1^l=F_2^l=F^l$，$\frac{hu}{hl}=\frac{F^l}{2\sqrt{2}F^u}$

时，有

图18　棋盘形四角錐网架

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 对 & a_{22} & a_{23} \\ & 称 & a_{33}\end{bmatrix}\begin{Bmatrix} u \\ v \\ w\end{Bmatrix}=\begin{Bmatrix} -x \\ -y \\ q\end{Bmatrix} \qquad (2.2.25)$$

式中 $a_{11}=d^l\left(\frac{2\sqrt{2}+n}{n}\right)\frac{\partial^2}{\partial x^2}+d^l\frac{\partial^2}{\partial y^2}$，$a_{12}=2d^l\frac{\partial^2}{\partial x\partial y}$，

$$a_{13}=-3d^lh_l\frac{\partial^3}{\partial x\partial y^2},$$

$$a_{22}=d^l\frac{\partial^2}{\partial x^2}+d^l\left(\frac{2\sqrt{2}+n}{n}\right)\frac{\partial^2}{\partial y^2},\quad a_{23}=-3d^lh_l\frac{\partial^3}{\partial x^2\partial y},$$

$$a_{33}=2\sqrt{2}d^l\frac{h^2}{2\sqrt{2}+n}\frac{\partial^4}{\partial x^4}+6d^lh_l^2\frac{\partial^4}{\partial x^2\partial y^2}+2\sqrt{2}d^l\frac{h^2}{2\sqrt{2}+n}\frac{\partial^4}{\partial y^4} \qquad (2.2.26)$$

其中 $d^l=\frac{EF^l}{2\sqrt{2}a}$，$h_l=\frac{2\sqrt{2}h}{2\sqrt{2}+n}$

因a_{13}、a_{23}中的系数不可能为零，所以这种网架的u、v与w也是耦合的。

若$x=y=0$，可引入辅助函数ϕ，并设

$$\left.\begin{aligned} u&=-\frac{3nh_l}{2\sqrt{2}+n}\frac{\partial^5\phi}{\partial x^3\partial y^2}+3h_l\frac{\partial^5\phi}{\partial x\partial y^4},\ v=-\frac{3nh_l}{2\sqrt{2}+n}\frac{\partial^5\phi}{\partial x^2\partial y^3}+3h_l\frac{\partial^5\phi}{\partial x^4\partial y}\\ w&=\frac{\partial^4\phi}{\partial x^4}+\left[\frac{2\sqrt{2}+n}{n}-\frac{3n}{2\sqrt{2}+n}\right]\frac{\partial^4\phi}{\partial x^2\partial y^2}+\frac{\partial^4\phi}{\partial y^4}\end{aligned}\right\}\quad(2.2.27)$$

将上式代入（2.2.25）式，（2.2.25）式中的前两式变为恒等式，而第三个方程变为

$$\frac{\partial^8\phi}{\partial x^8}+\frac{\partial^8\phi}{\partial y^8}+\frac{2(6\sqrt{2}+n)}{n}\frac{\partial^8\phi}{\partial x^4\partial y^4}+\frac{2(\sqrt{2}-n)}{n}\left(\frac{\partial^8\phi}{\partial x^6\partial y^2}+\frac{\partial^8\phi}{\partial x^2\partial y^6}\right)=\frac{q}{B}\qquad(2.2.28)$$

式中　$B=\dfrac{EF^lh^2}{(2\sqrt{2}+n)a}$

三、考虑剪切变形时网架微分方程式的建立

1. 一般微分方程式的建立

根据夹层板理论的假定，在夹心中$\sigma_x=\sigma_y=\tau_{xy}=0$。夹心在$x$、$y$方向上的平衡方程为

$$\frac{\partial\tau_{xz}}{\partial z}=0\quad\frac{\partial\tau_{yz}}{\partial z}=0\qquad(3.1.1)$$

可见，τ_{xz}与τ_{yz}只是x、y的函数，即剪应力沿夹心厚度为均匀分布。

根据假定，只有夹心中的剪应力构成板的横向剪力Q_x和Q_y，故有

$$\tau_{xz}=\frac{Q_x}{h}\ ,\ \tau_{yz}=\frac{Q_y}{h}\qquad(3.1.2)$$

根据虎克定律，相应的剪应变为

$$\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}=\frac{Q_x}{C_1},\ \frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}=\frac{Q_y}{C_2}\qquad(3.1.3)$$

式中C_1及C_2分别为夹心在xz和yz平面内的剪切刚度，它们可通过将空间网格连续化，用类似于平面网格连续化的方法求得。

若设　$\phi_x=\dfrac{\partial w}{\partial x}-\dfrac{Q_x}{C_1},\ \phi_y=\dfrac{\partial w}{\partial y}-\dfrac{Q_y}{C_2}$　(3.1.4)

则　$Q_x=C_1\left(\dfrac{\partial w}{\partial x}-\phi_x\right),\ Q_y=C_2\left(\dfrac{\partial w}{\partial y}-\phi_y\right)$　(3.1.5)

上表层的应变为

$$\varepsilon^u_x=\frac{\partial u}{\partial x}+h_u\frac{\partial\phi_x}{\partial x},\ \varepsilon^u_y=\frac{\partial v}{\partial y}+h_u\frac{\partial\phi_y}{\partial y},\ \varepsilon^u_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+h_u\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\qquad(3.1.6)$$

下表层的应变为

$$\varepsilon^l_x=\frac{\partial u}{\partial x}-h_l\frac{\partial\phi_x}{\partial x},\ \varepsilon^l_y=\frac{\partial v}{\partial y}-h_l\frac{\partial\phi_y}{\partial y},\ \varepsilon^l_{xy}=\frac{\partial u}{\partial y}+\frac{yv}{\partial x}-h_l\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\qquad(3.1.7)$$

断面力表式与（2.1.5）式同。将上述有关各式代入（2.1.5）式中，可得

$$\left\{\begin{matrix} \mathbf{N} \\ \mathbf{M} \end{matrix}\right\} = \begin{bmatrix} \mathbf{P} & \mathbf{S} \\ \mathbf{S} & \mathbf{B} \end{bmatrix} \left\{\begin{matrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\theta} \end{matrix}\right\} \tag{3.1.8}$$

式中

$$\boldsymbol{\theta} = \left\{\begin{matrix} -\dfrac{\partial \phi_x}{\partial x} \\ -\dfrac{\partial \phi_y}{\partial y} \\ -\left(\dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x}\right) \end{matrix}\right\} \tag{3.1.9}$$

其余符号与前同。

显然，考虑剪切变形时板弯曲作用与膜作用分离的条件仍为（2.1.9）式。

而板的平衡微分方程式为

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} + X = 0, \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} + Y = 0 \tag{3.1.10}$$

$$\left.\begin{aligned} &\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0, \quad \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y = 0 \\ &\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0 \end{aligned}\right\} \tag{3.1.11}$$

将（3.1.5）、（3.18）式代入（3.1.10）、（3.1.11）式中，整理后可得

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ & a_{22} & a_{23} & a_{24} & a_{25} \\ \text{对} & & a_{33} & a_{34} & a_{35} \\ & & & a_{44} & a_{45} \\ & & \text{称} & & a_{55} \end{pmatrix} \begin{Bmatrix} u \\ v \\ \phi_x \\ \phi_y \\ w \end{Bmatrix} = \begin{Bmatrix} -X \\ -Y \\ 0 \\ 0 \\ q \end{Bmatrix} \tag{3.1.12}$$

式中　a_{11}、a_{12}、a_{22}与（2.1.12）式中的相同，

$$a_{13} = -s_{11}\frac{\partial^2}{\partial x^2} - 2s_{13}\frac{\partial^2}{\partial x \partial y} - s_{33}\frac{\partial^2}{\partial y^2},$$

$$a_{14} = -s_{13}\frac{\partial^2}{\partial x^2} - (s_{12} + s_{33})\frac{\partial^2}{\partial x \partial y} - s_{23}\frac{\partial^2}{\partial y^2} = a_{23}, \quad a_{15} = a_{25} = 0,$$

$$a_{24} = -s_{33}\frac{\partial^2}{\partial x^2} - 2s_{23}\frac{\partial^2}{\partial x \partial y} - s_{22}\frac{\partial^2}{\partial y^2},$$

$$a_{33} = b_{11}\frac{\partial^2}{\partial x^2} + 2b_{13}\frac{\partial^2}{\partial x \partial y} + b_{33}\frac{\partial^2}{\partial y^2} - C_1,$$

$$a_{34} = b_{13}\frac{\partial^2}{\partial x^2} + (b_{12} + b_{33})\frac{\partial^2}{\partial x \partial y} + b_{23}\frac{\partial^2}{\partial y^2}, a_{35} = C_1\frac{\partial}{\partial x}, a_{45} = C_2\frac{\partial}{\partial y},$$

$$a_{44} = b_{33}\frac{\partial^2}{\partial x^2} + 2b_{23}\frac{\partial^2}{\partial x \partial y} + b_{22}\frac{\partial^2}{\partial y^2} - C_2, a_{55} = -\left(C_1\frac{\partial^2}{\partial x^2} + C_2\frac{\partial^2}{\partial y^2}\right), \tag{3.1.13}$$

由上可知，若要u、v与ϕ_x、ϕ_y不耦合，a_{13}、a_{14}、a_{23}和a_{24}中的系数就必须全为零，即

$$\frac{h_u}{h_l}=\frac{D_{11}^l}{D_{11}^u}=\frac{D_{22}^l}{D_{22}^u}=\frac{D_{13}^l}{D_{13}^u}=\frac{D_{23}^l}{D_{23}^u}=\frac{D_{12}^l+D_{33}^l}{D_{12}^u+D_{33}^u}=\frac{D_{33}^l}{D_{33}^u} \tag{3.1.14}$$

2. 各种网架考虑剪切变形时的微分方程式

将各种网架的D^u及D^l代入一般方程式(3.1.12)，可得到下列各种网架的微分方程式

(1)两向正交正放网架、正放四角锥网架、正放抽空四角锥网架

$$A\frac{\partial^2 u}{\partial x^2}+X=0\ ,\quad A\frac{\partial^2 v}{\partial y^2}+Y=0 \tag{3.2.1}$$

$$B\frac{\partial^2\phi_x}{\partial x^2}+C\left(\frac{\partial w}{\partial x}-\phi_x\right)=0\ ,\quad B\frac{\partial^2\phi_y}{\partial y^2}+C\left(\frac{\partial w}{\partial y}-\phi_y\right)=0\ ,$$

$$C\left(\frac{\partial^2 w}{\partial x^2}-\frac{\partial\phi_x}{\partial x}\right)+C\left(\frac{\partial^2 w}{\partial y^2}-\frac{\partial\phi_y}{\partial y}\right)+q=0 \tag{3.2.2}$$

式中剪切刚度C为

$$C=\frac{EF_d\sin\varphi\cos^2\varphi}{a}\text{(两向正交正放网架)},\quad C=\frac{2EF_dC_xC_z^2}{a}\text{(正放四角锥网架)}$$

$$C=\frac{EF_dC_xC_z^2}{a}\text{(正放抽空四角锥网架)}$$

其中φ为斜杆与竖杆的夹角，C_xC_z为斜杆的方向余弦，F_d为斜杆面积。

当没有水平力作用时，只需求解方程式(3.2.2)。

若设

$$\left.\begin{aligned}&\phi_x=\left(1-\frac{B}{C}\frac{\partial^2}{\partial y^2}\right)\frac{\partial\omega}{\partial x},\quad \phi_y=\left(1-\frac{B}{C}\frac{\partial^2}{\partial x^2}\right)\frac{\partial\omega}{\partial y}\\&w=\left(1-\frac{B}{C}\frac{\partial^2}{\partial x^2}\right)\left(1-\frac{B}{C}\frac{\partial^2}{\partial y^2}\right)\omega\end{aligned}\right\} \tag{3.2.3}$$

并将(3.2.3)式代入(3.2.2)式，前两式变为恒等式，而第三式变为

$$\frac{\partial^4\omega}{\partial x^4}+\frac{\partial^4\omega}{\partial y^4}-\frac{B}{C}\left(\frac{\partial^6\omega}{\partial x^4\partial y^2}+\frac{\partial^6\omega}{\partial x^2\partial y^4}\right)=\frac{q}{B} \tag{3.2.4}$$

当$C\to\infty$时，(3.2.4)式就变为(2.2.2)式。

(2)两向正交斜放网架

$$\frac{A}{2}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+A\frac{\partial^2 v}{\partial x\partial y}+X=0\ ,\quad A\frac{\partial^2 u}{\partial x\partial y}+\frac{A}{2}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+Y=0 \tag{3.2.5}$$

$$\left.\begin{aligned}&B\left(\frac{\partial^2\phi_x}{\partial x^2}+\frac{\partial^2\phi_x}{\partial y^2}\right)+2B\frac{\partial^2\phi_y}{\partial x\partial y}+C\left(\frac{\partial w}{\partial x}-\phi_x\right)=0\ ,\\&2B\frac{\partial^2\phi_x}{\partial x\partial y}+B\left(\frac{\partial^2\phi_y}{\partial x^2}+\frac{\partial^2\phi_y}{\partial y^2}\right)+C\left(\frac{\partial w}{\partial y}-\phi_y\right)=0\\&C\left(\frac{\partial^2 w}{\partial x^2}-\frac{\partial\phi_x}{\partial x}\right)+C\left(\frac{\partial^2 w}{\partial y^2}-\frac{\partial\phi_y}{\partial y}\right)+q=0\end{aligned}\right\} \tag{3.2.6}$$

式中

$$C=EF_a sin\varphi cos^2\varphi/a$$

当没有水平力作用下，只需求解(3.2.6)式。

若设

$$\left.\begin{aligned}\phi_x&=\left(1-\frac{B}{C}\nabla^2\right)\frac{\partial\omega}{\partial x}+\frac{2B}{C}\frac{\partial^3\omega}{\partial x\partial y^2}\\ \phi_y&=\left(1-\frac{B}{C}\nabla^2\right)\frac{\partial\omega}{\partial y}+\frac{2B}{C}\frac{\partial^3\omega}{\partial x^2\partial y}\\ w&=\left(1-\frac{B}{C}\nabla^2-\frac{2B}{C}\frac{\partial^2}{\partial x\partial y}\right)\left(1-\frac{B}{C}\nabla^2+\frac{2B}{C}\frac{\partial^2}{\partial x\partial y}\right)\omega\end{aligned}\right\}\tag{3.2.7}$$

将(3.2.7)式代入(3.2.6)式，前两式变为恒等式，而第三式变为

$$\frac{\partial^4\omega}{\partial x^4}+6\frac{\partial^4\omega}{\partial x^2\partial y^2}+\frac{\partial^4\omega}{\partial y^4}-\frac{B}{C}\left(\frac{\partial^6\omega}{\partial x^6}-\frac{\partial^6\omega}{\partial x^4\partial y^2}-\frac{\partial^6\omega}{\partial x^2\partial y^4}+\frac{\partial^6\omega}{\partial y^6}\right)=\frac{q}{B}\tag{3.2.8}$$

(3)两向斜交斜放网架($\alpha=60°$)

$$\begin{aligned}&A\left(\frac{\partial^2u}{\partial x^2}+\frac{1}{3}\frac{\partial^2u}{\partial y^2}\right)+\frac{2}{3}A\frac{\partial^2v}{\partial x\partial y}+X=0,\\ &\frac{2}{3}A\frac{\partial^2u}{\partial x\partial y}+A\left(\frac{1}{3}\frac{\partial^2v}{\partial x^2}+\frac{1}{9}\frac{\partial^2v}{\partial y^2}\right)+Y=0\end{aligned}\tag{3.2.9}$$

$$\left.\begin{aligned}&B\left(\frac{\partial^2\phi_x}{\partial x^2}+\frac{1}{3}\frac{\partial^2\phi_x}{\partial y^2}\right)+\frac{2}{3}B\frac{\partial^2\phi_y}{\partial x\partial y}+C\left(\frac{\partial\omega}{\partial x}-\phi_x\right)=0,\\ &\frac{2}{3}B\frac{\partial^2\phi_x}{\partial x\partial y}+B\left(\frac{1}{3}\frac{\partial^2\phi_y}{\partial x^2}+\frac{1}{9}\frac{\partial^2\phi_y}{\partial y^2}\right)+C\left(\frac{\partial w}{\partial y}-\phi_y\right)=0\\ &C\left(\frac{\partial^2w}{\partial x^2}-\frac{\partial\phi_x}{\partial x}\right)+C\left(\frac{\partial^2w}{\partial y^2}-\frac{\partial\phi_y}{\partial y}\right)+q=0\end{aligned}\right\}\tag{3.2.10}$$

式中

$$C=EF_d sin\varphi cos^2\varphi/a$$

(4)三向网架、三角锥网架、抽空三角锥网架

$$\begin{aligned}&A\left(\frac{\partial^2u}{\partial x^2}+\frac{1}{3}\frac{\partial^2u}{\partial y^2}\right)+\frac{2}{3}A\frac{\partial^2v}{\partial x\partial y}+X=0,\\ &\frac{2}{3}A\frac{\partial^2u}{\partial x\partial y}+A\left(\frac{1}{3}\frac{\partial^2v}{\partial x^2}+\frac{\partial^2v}{\partial y^2}\right)+Y=0\end{aligned}\tag{3.2.11}$$

$$\left.\begin{aligned}&B\left(\frac{\partial^2\phi_x}{\partial x^2}+\frac{1}{3}\frac{\partial^2\phi_x}{\partial y^2}\right)+\frac{2}{3}B\frac{\partial^2\phi_y}{\partial x\partial y}+C\left(\frac{\partial w}{\partial x}-\phi_x\right)=0,\\ &\frac{2}{3}B\frac{\partial^2\phi_x}{\partial x\partial y}+B\left(\frac{1}{3}\frac{\partial^2\phi_y}{\partial x^2}+\frac{\partial^2\phi_y}{\partial y^2}\right)+C\left(\frac{\partial w}{\partial y}-\phi_y\right)=0\\ &C\left(\frac{\partial^2w}{\partial x^2}-\frac{\partial\phi_x}{\partial x}\right)+C\left(\frac{\partial^2w}{\partial y^2}-\frac{\partial\phi_y}{\partial y}\right)+q=0\end{aligned}\right\}\tag{3.2.12}$$

式中

$$C=\frac{\sqrt{3}EF_d sin\varphi cos^2\varphi}{l}\text{(三向网架)}\quad C=\frac{2EF_dC_xC_z^2}{\sqrt{3}l}\text{(三角锥网架)}$$

$$C=\frac{EF_dC_xC_z^2}{\sqrt{3}l}\text{(抽空三角锥网架)}$$

当没有水平力作用时，只需求解(3.2.12)式。

若设

$$\phi_x=\frac{\partial\omega}{\partial x}+\frac{\partial f}{\partial y},\quad \phi_y=\frac{\partial\omega}{\partial y}-\frac{\partial f}{\partial x},\quad w=\left(1-\frac{B}{C}\nabla^2\right)\omega \tag{3.2.13}$$

将上式代入(3.2.12)式，可得

$$\frac{B}{3C}\nabla^2 f-f=0,\qquad \nabla^2\nabla^2\omega=\frac{q}{B} \tag{3.2.14}$$

其中

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

(5)斜放四角锥网架、星形四角锥网架及棋盘形四角锥网架

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 \\ & a_{22} & a_{23} & a_{24} & 0 \\ \text{对} & & a_{33} & a_{34} & a_{35} \\ & & & a_{44} & a_{45} \\ & & \text{称} & & a_{55} \end{pmatrix}\begin{Bmatrix} u \\ v \\ \phi_x \\ \phi_y \\ w \end{Bmatrix}=\begin{Bmatrix} -X \\ -Y \\ 0 \\ 0 \\ q \end{Bmatrix} \tag{3.2.15}$$

对于斜放四角锥网架、星形四角锥网架

a_{11}、a_{12}、a_{22}与(2.2.22)式中的相同，$a_{13}=d^uh_u\dfrac{\partial^2}{\partial y^2}$，$a_{14}=2d^uh_u\dfrac{\partial^2}{\partial x\partial y}$，

$$a_{14}=a_{23},\ a_{24}=d^uh_u\frac{\partial^2}{\partial x^2},\ a_{33}=\frac{2nd^uh^2}{\sqrt{2}(1+\sqrt{2}n)}\frac{\partial^2}{\partial x^2}+d^uh_u^2\frac{\partial^2}{\partial y^2}-C,$$

$$a_{34}=2d^uh_u^2\frac{\partial^2}{\partial x\partial y},\ a_{35}=C\frac{\partial}{\partial x},\ a_{44}=d^uh_u^2\frac{\partial^2}{\partial x^2}+\frac{2nd^uh^2}{\sqrt{2}(1+\sqrt{2}n)}\frac{\partial^2}{\partial y^2}-C,$$

$$a_{45}=C\frac{\partial}{\partial y},\quad a_{55}=-C\nabla^2 \tag{3.2.16}$$

其中

$$C=\frac{EF_dC_xC_z^2}{a}\text{(斜放四角锥)},\qquad C=\frac{EF_d sin\varphi cos^2\varphi}{a}\text{(星形四角锥)}$$

对于棋盘形四角锥网架

a_{11}、a_{12}、a_{22}与(2.2.26)式中的相同，$a_{13}=-d^lh_l\dfrac{\partial^2}{\partial y^2}$，

$$a_{14}=a_{23}=-2d^l h_l\frac{\partial}{\partial x\partial y},\ a_{33}=2\sqrt{2}\,d^l\frac{h^2}{(2\sqrt{2}+n)}\frac{\partial^2}{\partial x^2}+d^l h_l^2\frac{\partial^2}{\partial y^2}-C$$

$$a_{24}=-d^l h_l\frac{\partial^2}{\partial x^2},\quad a_{34}=d^l h_z^2\frac{\partial^2}{\partial x\partial y},\quad a_{35}=C\frac{\partial}{\partial x}$$

$$a_{44}=d^l h_l^2\frac{\partial}{\partial x^2}+2\sqrt{2}\,d^l\frac{h^2}{(2\sqrt{2}+n)}\frac{\partial^2}{\partial y^2}-C,\ a_{45}=C\frac{\partial}{\partial y},\quad a_{55}=-C\nabla^2$$

其中　　(3.2.17)

$$C=\frac{EF_d C_x C_z^2}{a}$$

显然，在(3.2.15)式中a_{13}、a_{14}、a_{23}及a_{24}不可能为零，因此u、v与ϕ_x、ϕ_y是耦合的。在求解时可引入一辅助函数ϕ，将(3.2.15)式合并为一个关于ϕ的10阶偏微分方程式。

四、某些网架位移及杆件内力的计算

对于上述各种形式网架的微分方程，可用求解平板或夹层板的现有方法来进行求解。但对于两向正交斜放网架、两向斜交斜放网架及三向网架等必须考虑剪切变形影响，才能得到精度较好的结果。应当注意，某些网格(例如正交正放网架等)所构成的表层，不能承受平面内的剪力，这是与真实夹层板的表层不同之处。正是由于假想夹层板与真夹层板的这个区别，使得在某些边界条件下，真夹层板有解而假想夹层板可能没有解。

为便于应用起见，下面给出某些网架的位移及杆件内力的计算公式。

1.周边简支的两向正交正放网架、正放四角锥网架及正放抽空四角锥网架在均布荷载作用下的计算公式

当$B_1=B_2=B$时，挠度w的公式为

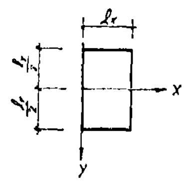


图　19

$$w=\frac{q}{24B}(x^4-2l_x x^3+l_x^3 x)-\frac{4ql_x^4}{\pi^5 B}\times\sum_{m=1,3}^{\infty}\left\{\frac{F_1(\beta_m)F_1(\xi_m)+4F_3(\beta_m)F_3(\xi_m)}{m^5[F_1^2(\beta_m)+4F_3^2(\beta_m)]}\right\}\sin\frac{m\pi x}{l_x}\qquad(4.1.1)$$

上弦杆内力公式

$$N_x^u=\frac{ax(x-l_x)q}{2h}+\frac{4al_x^2 q}{\pi^3 h}\sum_{m=1,3}^{\infty}\frac{F_1(\beta_m)F_1(\xi_m)+4F_3(\beta_m)F_3(\xi_m)}{m^3[F_1^2(\beta_m)+4F_3^2(\beta_m)]}\sin\frac{m\pi x}{l_x}$$

$$N_y^u=\frac{8aql_x^2}{\pi^3 h}\sum_{m=1,3}^{\infty}\frac{F_1(\beta_m)F_3(\xi_m)-F_3(\beta_m)F_1(\xi_m)}{m^3[F_1^2(\beta_m)+4F_3^2(\beta_m)]}\sin\frac{m\pi x}{l_x}\qquad(4.1.2)$$

下弦杆内力公式

$$N_x^l = -N_x^u \ , \ N_y^l = -N_y^u \text{（正交正放，正放四角锥网架）}$$

$$N_x^l = -2N_x^u \ , \ N_y^l = -2N_y^u \text{（正放抽空四角锥网架）}$$

横向剪力公式

$$Q_x = -\frac{q(2x-l_x)}{2} - \frac{4ql_x}{\pi^2}\sum_{m=1,3}^{\infty}\frac{F_1(\beta_m)F_1(\xi_m)+4F_3(\beta_m)F_3(\xi_m)}{m^2[F_1^2(\beta_m)+4F_3^2(\beta_m)]}\cos\frac{m\pi x}{l_x}$$

$$Q_y = -\frac{4\sqrt{2}ql_x}{\pi^2}\sum_{m=1,3}^{\infty}\frac{F_1(\beta_m)F_2(\xi_m)+4F_3(\beta_m)F_4(\xi_m)}{m^2[F_1^2(\beta_m)+4F_3^2(\beta_m)]}\sin\frac{m\pi x}{l_x} \tag{4.1.3}$$

网架腹杆的内力可通过横向剪力Q_x、Q_y求出。

式中$\alpha_m=\frac{\sqrt{2}}{2}\frac{m\pi}{l_x}$，$\beta_m=\frac{\sqrt{2}}{4}\frac{m\pi l_y}{l_x}$，$\xi_m=\alpha_m y$，$F_1(\xi_m)$，$F_2(\xi_m)$，$F_3(\xi_m)$，$F_4(\xi_m)$为克雷洛夫函数，其表达式为

$$F_1(\xi_m)=ch(\xi_m)cos(\xi_m),\quad F_2(\xi_m)=\frac{1}{2}[ch(\xi_m)sin(\xi_m)+sh(\xi_m)cos(\xi_m)]$$

$$F_3(\xi_m)=\frac{1}{2}sh(\xi_m)sin(\xi_m),\quad F_4(\xi_m)=\frac{1}{4}[ch(\xi_m)sin(\xi_m)-sh(\xi_m)cos(\xi_m)]$$

这类函数已有现成的表格可查[8]。

当$B_1\neq B_2$时在上述公式中应用α'_m、β'_m、ξ'_m来代替α_m、β_m、ξ_m。而

$$\alpha'_m=\frac{\sqrt{2}}{2}\eta\frac{m\pi}{l_x}\ ,\ \beta'_m=\alpha'_m\frac{l_y}{2}\ ,\ \xi'_m=\alpha'_m y\ (\eta^4=\frac{B_1}{B_2}).$$

2．斜放四角锥网架及星形四角锥网架的近似计算

根据这类网架总的受力特性，可近似地按正交正放网架来分析。实际的计算结果说明，这种简化分析法能保证工程实践上所要求的精度。

下面给出按此法所得到的位移及内力计算公式。

挠度w按(4.1.1)式计算，其中

$$B=\frac{\sqrt{2}F^uF^l}{(\sqrt{2}F^u+F^l)}\frac{Eh^2}{l}\quad \text{（}l\text{为下弦网格的边长）}$$

上弦杆内力

$$N_{12}^u=N_{13}^u=\frac{qlx(x-l_x)}{4\sqrt{2}h}+\frac{2lql_x^2}{\sqrt{2}\pi^3h}\sum_{m=1,3}^{\infty}\frac{F_1(\beta_m)F_1(\xi_m)+4F_3(\beta_m)F_3(\xi_m)+2F_1(\beta_m)F_3(\xi_m)-2F_3(\beta_m)F_1(\xi_m)}{m^3[F_1^2(\beta_m)+4F_3^2(\beta_m)]}\sin\frac{m\pi x}{l_x} \tag{4.2.1}$$

下弦杆内力

$$N_x^l=\frac{-ql(x-l_x)x}{2h}-\frac{4lql_x^2}{\pi^3h}\sum_{m=1,3}^{\infty}\frac{F_1(\beta_m)F_1(\xi_m)+4F_3(\beta_m)F_3(\xi_m)}{m^3[F_1^2(\beta_m)+4F_3^2(\beta_m)]}\sin\frac{m\pi x}{l_x}$$

$$N_y^l=-\frac{8qll_x^2}{\pi^3h}\sum_{m=1,3}^{\infty}\frac{F_1(\beta_m)F_3(\xi_m)-F_3(\beta_m)F_1(\xi_m)}{m^3[F_1^2(\beta_m)+4F_3^2(\beta_m)]}\sin\frac{m\pi x}{l_x} \tag{4.2.2}$$

五、计算结果的比较

1. 两向正交正放网架

几何尺寸 $l_x=l_y=48^m, a=3^m, h=3.6^m$；杆件截面积 $F^u=12^{cm^2}, F^l=F_{竖}=8^{cm^2}$，$F_d=6^{cm^2}$；荷载 $q=0.2T/m^2$。

上弦杆内力的比较(T)

本文公式计算结果（级数取二项）	重三角级数取100项的结果（考虑剪切变形）	重三角级数结果（取100项，不考虑剪切变形）	矩阵位移法的结果
−29.62	−29.4	−29.61	−29.48
−28.20	−27.8	−28.20	−28.00
−27.36	−27.15	−27.40	−27.50
−23.26	−22.95	−23.35	−23.45
−20.91	−21.00	−21.40	−21.02
−14.18	−13.90	−14.10	−15.02
−11.30	−11.80	−11.30	−10.46
− 7.76	− 7.76	− 7.77	− 8.8
− 5.76	− 5.93		− 3.62
−16.85	−16.90		−16.65
− 5.98	− 5.94		− 5.75
− 4.77	− 4.99		− 4.80
− 2.20	− 2.06		− 2.12

支座处斜杆最大内力(T)

13.27（本文公式）12.90（重三角级数）13.75（矩阵位移法）

在对称轴上某点的挠度(cm)

18.55（本文公式）22.22（矩阵位移法）

2. 斜放四角锥网架

几何尺寸 $l_x=l_y=36^m, h=2.5^m, l=3^m$；荷载 $q=0.154T/m^2$。

下弦杆内力(T)

本文公式	假想弯矩法	矩阵位移法	
		Ⅰ	Ⅱ
18.21	17.10	18.80	18.01
15.58	13.70	15.28	14.52
14.57	14.65	15.04	14.95
12.79	10.40	11.87	11.16
11.33	12.05	9.38	10.17
8.68	5.90	6.90	6.38
7.01	8.32	4.60	6.11
2.39	3.29	-1.77	1.86
12.67	11.80	12.23	12.07

上弦杆内力(T)

本文公式	假想弯矩法	矩阵位移法	
		Ⅰ	Ⅱ
-12.99	-12.30	-8.86	-5.77
-10.73	-10.50	-9.62	-6.13
-2.02	-2.22	-2.57	-1.70
-5.58	-5.87	-6.24	-4.16
-12.62	-12.30	-9.64	-6.15
-8.56	-8.95	-10.75	-8.57
-9.58	-8.90	-9.78	-7.57
-7.52	-7.50	-4.10	-0.93
-8.12	-7.50	-9.19	-7.73

边界Ⅰ：$w=0$，边界Ⅱ：$w=0$，及沿周边切向位移为零.

六、结　语

1．对网架结构来说，这样的连续化处理是近似的，只有当网格尺寸远小于整个网架尺寸，以及荷载变化较均匀时才是合适的。

2．当网架的节点及杆件数很大，使得计算机分析时间很长，或在初步设计及优化设计时，按连续化模型进行分析仍不失为一种有效的方法。

3．网架的夹层板模型，能从总体上反映网架结构的受力和变形的特点，这样就能对它们的相对有效性作出定量的估计。同时，本文的连续化方法对于类似的离散化结构有普遍意义，并特别适用于分析屋面板与网架合一的板架结构。

4．对于两向正交正放网架，正放四角锥网架及正放抽空四角锥网架，本文所给出的公式有足够的精度，与精确解相比，在网架大部分区域杆件内力误差5%以内。而且，剪切变形对杆件内力影响不大，但对挠度影响较大。不考虑剪切变形的挠度偏小，一般可乘以系数1.2—1.3进行修正。

5．对于斜放四角锥网架，本文所给出的简化计算公式有一定的精度。对下弦杆内力而言，计算结果与矩阵位移法的较为接近；对上弦杆内力而言，计算结果与假想弯矩法的较为接近。这表明简化计算公式的精度与假想弯矩法相当。

6．本文给出的计算公式收敛很快，一般取级数前两项就能达到精确解的结果。所以，在网架的优化设计中采用本文公式进行结构分析是合适的。

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Analysis of Double Layer Grid as Equivalent Sandwich Plate

Luo En

Abstract

In this paper double layer grid is regarded as sandwich plate. According to the theory of sandwich plates, the general differential equations of double layer grid are established. The differential equations of various types of double layer grids are also given in the paper. Computed results exhibit good agreement with those of the matrix displacement method. Because of the good accuracy of formulae and the rapid convergency of the series, the formulae given in the paper are especially applicable to optimum design of double layer grid.