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The Exact Distribution of the Estimator of Parameters in the SUR Model

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Abstract

We derive the exact distribution of the estimator of parameters in the SUR model with the distubance of the third kind elliptically contoured distribution, thus generalizing the result of Phillips [1].

Keywords the SUR model, exact distribution, fractional operator elliptically contoured distribution.

1. Introduction

Let the multivariate linear model be in the form:

$$y_t = Ax_t + u_t \qquad (t=1,\cdots,T) \tag{1}$$

where y_t is an $n\times 1$ vector of endogenous variables, x_t is an $m\times 1$ vector of nonrandom exogenous variables, and u_t is an $n\times 1$ random vector. We also write (1) as

$$Y' = AX' + U' \tag{2}$$

where the data matrices are assembled in columns as $Y'=[y_1,\cdots, y_T]$; and we assume that X has full rank m and U has the third kind elliptically contoured distribution, i. e. $U\sim LEC_{T\times(n+1)}(0, \Sigma\otimes I; g)$. The probability density function of U is

$$p.d.f(U) = (2\pi)^{-\frac{1}{2}(n+1)T}\det(\Sigma)^{-\frac{1}{2}}g(\mathrm{tr}U(\Sigma\otimes I)^{-1}U') \tag{3}$$

where g is a certain apropriate function.

The coefficient matrix A in (2) is assumed to be parameterized in the form

$$\mathrm{vec}(A) = Sa - s \tag{4}$$

where vec ( ) denotes vectorization by rows, S is an nmxq matrix whose elements are known constants and whose rank is q, and s is a vector of known constants. In (4) a is taken as the $(q\times 1)$ vector of basic parameters. The model given by (2) and (4) becomes the SUR model, when $s=0$, and S is a block diagonal selector matrix.

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Zellner[2] developed a two-stage GLS estimator for the parameters in the SUR model. This procedure has been used in many empirical applications. From (2), we have

$$\mathrm{vec}(Y')=(I\otimes Y)\mathrm{vec}(A)+\mathrm{vec}(U')=(I\otimes X)S\alpha-(I\otimes X)s+\mathrm{vec}(U').$$

Because the covariance matrix of U is a certain constant multiple of Σ[3], the GLS estimator of α is given by

$$\hat{\alpha}=\{S'(\Sigma^{-1}\otimes X'X)S\}^{-1}\{S'(\Sigma^{-1}\otimes X')\mathrm{vec}(Y')+S'(\Sigma^{-1}\otimes X'X)s\} \quad (5)$$

The two-stage estimator of α is obtained by replacing Σ in (5) by an estimate that is based on the residuals of a preliminary least squares regression on (2). We take the estimate

$$\hat{\Sigma}=kY'(I-P_X)Y \quad (6)$$

where $P_X=X(X'X)^{-1}X'$, and k is a certain constant, from an unrestricted regression. The corresponding two-stage estimate of α will denote by α^*. The error in this estimate satisfies:

$$\alpha^*-\alpha=\{S'(\hat{\Sigma}^{-1}\otimes X'X)S\}^{-1}\{S'(\hat{\Sigma}^{-1}\otimes X')\mathrm{vec}(U')\} \quad (7)$$

Phillips (1985) derived the exact distribution of α^*, when U is normal, i.e. $U\sim N(0,\ \Sigma\otimes I)$. In this paper we will derive the exact distribution of α^*, when U has the third kind elliptically contoured distribution.

2. Fractional Matrix Calculus

For deriving the exact distribution of α^*, this section extends the the theory of fractional operators in differential calculus to matrix spaces.

Definition If f is a complex analytic function of X and α is a complex number for which $\mathrm{Re}(\alpha)>(n-1)/2$ we define the fractional matrix operator $(\det\partial X)^{-\alpha}$ by the integral

$$(\det\partial X)^{-\alpha}f(X)=\frac{1}{\Gamma_n(\alpha)}\int_{S>0}f(X-S)(\det S)^{\alpha-(n+1)/2}\mathrm{d}S \quad (8)$$

when it exists. X is an $n\times n$ complex matrix, and the integral is taken over the set of positive definte matrices $S>0$.

It is convenient to work with the adjoint of the matrix operator ∂X, which we will write in the form $\partial X_a=\mathrm{adj}(\partial/\partial X)$. The fractional calculus can be defined as

$$(\det\partial X_a)^{-\alpha}f(X)=\frac{1}{\Gamma_n(\alpha)}\int_{S>0}\{\mathrm{etr}(-\partial X_aS)f(X)\}(\det S)^{\alpha-(n+1)/2}\mathrm{d}S \quad (9)$$

$$\mathrm{Re}(\alpha)>(n-1)/2$$

provided the integral exists.

Let $S_{mn\times q}$ and $M_{m\times m}$ be constant matrices, then, from Phillips[1], we have

$$\{\det(S'(\partial X_a\otimes M)S)\}^{\mu}\operatorname{etr}(AX)=\operatorname{etr}(AX)\{\det(S'(A_a\otimes M)S)\}^{\mu} \tag{10}$$

where μ is an arbitrary complex, number $\operatorname{Re}(\mu)>-(n-1)/2, A_a=A^{-1}/|A|$,

$$\{\det(S'(\partial X_a\Sigma\partial X_a\otimes M)S)\}^{\mu}\operatorname{etr}(AX)$$
$$=\operatorname{etr}(AX)\{\det(S'(A_a\Sigma A_a\otimes M)S)\}^{\mu} \tag{11}$$

$$\{g'(S'(\partial X_a\otimes M)S)(S'(\partial X_a\Sigma\partial X_a\otimes M)S)_a(S'(\partial X_a\otimes M)S)g\}^{j}\operatorname{etr}(AX)$$
$$=\operatorname{etr}(AX)\{g'(S'(A_a\otimes M)S)(S'(A_a\Sigma A_a\otimes M)S)_a(S'(A_a\otimes M)S)g\}^{j} \tag{12}$$

where j is a positive integer, and g is an $nT\times 1$ vector.

Lemma Let

$$H_j=\int_{D>0} g(\operatorname{tr}\Sigma^{-1}D)\ (\det D)^{\frac{1}{2}(T-m-n-1)}\{e'(S'(D_a\otimes M)S)$$
$$(S'(D_a\otimes M)S)_a(S'(D_a\otimes M)S)e\}^{j}\det(S'(D_a\otimes M)S)$$
$$\det(S'(D_a\Sigma D_a\otimes M)S)\}^{-j-1/2}\,\mathrm{d}D$$

where g is an arbitrary function, which provides the existence of the integral. We assume that X is a matrix, $X_a=X^{-1}/|X|$, S is an $nm\times q$ constant matrix whose rank is q, and $e_{q\times 1}$ is a vector.

Then

$$H_j=\{e'[S'(\partial W_a\otimes M)S][S'(\partial W_a\Sigma\partial W_a\otimes M)S]_a[S'(\partial W_a\otimes M)S]e\}^{j}$$
$$\cdot\det[S'(\partial W_a\otimes M)S]\{\det[S'(\partial W_a\Sigma\partial W_a\otimes M)S]\}^{-j-1/2}$$
$$\cdot\int_{D>0} g(\operatorname{tr}\Sigma^{-1}D)\operatorname{etr}(WD)\ (\det D)^{\frac{1}{2}(T-m-n-1)}\,\mathrm{d}D\Big|_{W=0} \tag{13}$$

If $G(\partial W_a)\triangleq[S'(\partial W_a\otimes M)S]^{-1}[S'[\partial W_a\Sigma\partial W_a\otimes M)S][S'(\partial W_a\otimes M)S]^{-1}$,

then $$H_j=\{e'[G(\partial W_a)]^{-1}e\}^{j}[\det G(\partial W_a)]^{-\frac{1}{2}}\int_{D>0} g(\operatorname{tr}\Sigma^{-1}D)\operatorname{etr}(WD)\cdot$$
$$\cdot(\det D)^{\frac{T-m-n-1}{2}}\,\mathrm{d}D\Big|_{W=0}.$$

Proof The right-hand side of (13) can be written as

$$\int_{D>0} g(\operatorname{tr}\Sigma^{-1}D)\ (\det D)^{\frac{1}{2}(T-m-n-1)}\{e'[S'(\partial W_a\otimes M)S][S'(\partial W_a\Sigma\partial W_a\otimes M)S]_a$$
$$\cdot[S'(\partial W_a\otimes M)S]e\}^{j}\det[S'(\partial W_a\otimes M)S]\cdot\{\det[s'(\partial W_a\Sigma\partial W_a\otimes M)S]\}^{-j-1/2}$$
$$\cdot\operatorname{etr}(DW)\mathrm{d}D|_{W=0}$$

$$\overset{(11)}{=}\int_{D>0} g(\operatorname{tr}\Sigma^{-1}D)(\det D)^{\frac{1}{2}(T-m-n-1)}\{e'[S'(\partial W_a\otimes M)S]\cdot$$
$$\cdot[S'(\partial W_a\Sigma\partial W_a\otimes M)S]_a\cdot[S'(\partial W_a\otimes M)S]e\}^{j}\det[S'(\partial W_a\otimes M)S]\operatorname{etr}(DW).$$
$$\cdot\{\det[S'(D_a\Sigma D_a\otimes M)S]\}^{-j-1/2}\,\mathrm{d}D|_{W=0}$$

$$\overset{(10)}{=}\int_{D>0} g(\operatorname{tr}\Sigma^{-1}D)\ (\det D)^{\frac{1}{2}(T-m-n-1)}\{e'[S'(\partial W_a\otimes M)S]\cdot$$
$$\cdot[S'(\partial W_a\Sigma\partial W_a\otimes M)S]_a\cdot[S'(\partial W_a\otimes M)S]e\}^{j}\operatorname{etr}(DW)\det[S'(D_a\otimes M)S]\cdot$$
$$\cdot\{\det[S'(D_a\Sigma D_a\otimes M)S]\}^{-j-1/2}\mathrm{d}D|_{W=0}$$

(12) H_i.

The proof is complete.

3. The Exact Distribution of α*

In this section, we derive the exact distribution of α^* using the result of section 2.

Let $M=T^{-1}X'X$, $p=\text{vec}(U'X/T)$, $D=Y'(I-P_x)Y$. From (7), we have

$$\alpha^*-\alpha=[S'(D^{-1}\otimes X'X)S]^{-1}[S'(D^{-1}\otimes X')\text{vec}(U')]\triangleq e(p,D)$$

Since $p=(I\otimes X/T)\text{vec}(U')$, p has the third kind elliptically contoured distribution, and

$$\text{p.d.f}(p)=(2\pi)^{-nT/2}[\det(\Sigma\otimes M/T)]^{-\frac{1}{2}}g(p'(\Sigma\otimes M/T)p)$$

Set $B=[S'(D^{-1}\otimes M)S]^{-1}[S'(D^{-1}\otimes I)]$, then

$$\alpha^*-\alpha=e(p,D)=\{S'(D^{-1}\otimes X'X)S\}^{-1}\{S'(D^{-1}\otimes I)\text{vec}(U'X/T)\}=Bp \quad (14)$$

Since p has the third kind elliptically contoured distribution, the conditional pdf of e given D is

$$\text{p.d.f}(e|D)=(2\pi)^{-q/2}T^{q/2}[\det(B\Sigma\otimes M)B')]^{-\frac{1}{2}}g(Te'[B(\Sigma\otimes M)B']^{-1}e) \quad (15)$$

The matrix D is generalized central Wishart with the density function given by [4]

$$\text{p.d.f}(D)=\frac{2^{-\frac{1}{2}(T-m)n}}{\Gamma_n\left(\frac{T-m}{2}\right)}(\det\Sigma)^{-\frac{1}{2}(T-m)}(\det D)^{\frac{1}{2}(T-m-n-1)}g(\text{tr}\Sigma^{-1}D) \quad (16)$$

It follows that the unconditional pdf of e is given by the integral:

$$\text{p.d.f.}(e)=\frac{2^{-\frac{1}{2}n(T-m)}T^{\frac{1}{2}q}(2\pi)^{-\frac{1}{2}q}}{\Gamma_n\left(\frac{T-m}{2}\right)(\det\Sigma)^{\frac{1}{2}(T-m)}}\int_{D>0}\frac{g(Te'(B(\Sigma\otimes M)B')^{-1}e)}{(\det(B(\Sigma\otimes M)B')^{\frac{1}{2}}}$$

$$\cdot g(\text{tr}\Sigma^{-1}D)(\det D)^{\frac{1}{2}(T-m-n-1)}\text{d}D \quad (17)$$

Suppose that g is Taylor expansible on $(-\infty,+\infty)$, and

$$(2\pi)^{-\frac{1}{2}q}g(Te'(B(\Sigma\otimes M)B')^{-1}e)=\sum_{j=0}^{\infty}\frac{b_j}{j!}(e'(B\Sigma\otimes M)B')^{-1}e)^j$$

We now decompose the matrix

$$B(\Sigma\otimes M)B'=[S'(D_a\otimes M)S]^{-1}[S'(D_a\Sigma D_a\otimes M)S][S'(D_a\otimes M)S]^{-1}$$

Then (17) can be written in the following form:

$$\text{p.d.f}(e)=\frac{2^{-\frac{1}{2}n(T-m)}T^{\frac{1}{2}q}}{\Gamma_n\left(\frac{T-m}{2}\right)(\det\Sigma)^{\frac{1}{2}(T-m)}}\sum_{j=0}^{\infty}\frac{b_jTj}{j!}\int_{D>0}g(\text{tr}\Sigma^{-1}D)$$

$$\cdot(\det D)^{\frac{1}{2}(T-m-n-1)}\{e'[S'D_a\otimes M)S][S'(D_a\Sigma D_a\otimes M)S]_a\cdot[S'(D_a\otimes M)S]e\}^j$$

$$\cdot \det[S'(D_a\otimes M)S]\{\det[S'(D_a\Sigma D_a\otimes M)S]\}^{-j-\frac{1}{2}}\mathrm{d}D \quad (18)$$

From the Lemma in section 2, we have

$$\text{p.d.f}(e)=\frac{2^{-\frac{1}{2}n(T-m)}T^{\frac{1}{2}q}}{\Gamma_n\left(\frac{T-m}{2}\right)(\det\Sigma)^{\frac{1}{2}(T-m)}}\sum_{j=0}^{\infty}\frac{b_jT^j}{j!}\{e'(G(\partial W_a))^{-1}e\}^j$$

$$\cdot(\det G(\partial W_a))^{-\frac{1}{2}}\int_{D>0}g(\mathrm{tr}\Sigma^{-1}D)(\det D)^{\frac{1}{2}(T-m-n-1)}\mathrm{etr}(WD)\mathrm{d}D\Big|_{W=0}$$

$$=\frac{2^{-\frac{1}{2}n(T-m)-\frac{1}{2}q}T^{\frac{1}{2}q}\pi^{-\frac{1}{2}q}g\{Te'G(\partial W_a)^{-1}e\}}{\Gamma_n\left(\frac{T-m}{2}\right)(\det\Sigma)^{\frac{1}{2}(T-m)}[\det G(\partial W_a)]^{\frac{1}{2}}}$$

$$\cdot\int_{D>0}g(\mathrm{tr}\Sigma^{-1}D)\,\mathrm{etr}(WD)(\det D)^{\frac{1}{2}(T-m-n-1)}\mathrm{d}D\Big|_{W=0} \quad (19)$$

since $(2\pi)^{-\frac{1}{2}q}g(h'\frac{\partial}{\partial x}h)=\sum_{j=0}^{\infty}\frac{b_j}{j!}(h'\frac{\partial}{\partial x}h)^j$, for arbitrary $q\times 1$ vector h.

Next, we need to evaluate the integral

$$P\triangleq\int_{D>0}g(\mathrm{tr}\Sigma^{-1}D)\,\mathrm{etr}(WD)(\det D)^{\frac{1}{2}(T-m-n-1)}\mathrm{d}D$$

Since $W_{n\times n}$ and $D_{n\times n}$ are all symmetric matrices, we have the zonal polynomial[5]

$$\mathrm{etr}(WD)=\sum_{j=0}^{\infty}\frac{1}{j!}\sum_{\kappa}C_{\kappa}(WD) \quad (20)$$

where the inner summation is over all partitions κ of the integer and $\kappa=(k_1,\cdots,k_n)$ is a partition of j into not more than m parts. Let $H\triangleq\Sigma^{-\frac{1}{2}}D\Sigma^{-\frac{1}{2}}$, $\tilde{W}\triangleq\Sigma^{\frac{1}{2}}W\Sigma^{\frac{1}{2}}$, then from Lemma6 [4] we have

$$P=\int_{H>0}g(\mathrm{tr}H)\mathrm{etr}(H\tilde{W})(\det H)^{\frac{1}{2}(T-m-n-1)}(\det\Sigma)^{\frac{1}{2}(T-m)}\mathrm{d}H$$

$$=\sum_{j=0}^{\infty}\frac{1}{j!}\sum_{\kappa}\int_{H>0}g(\mathrm{tr}H)\cdot C_{\kappa}(H\tilde{W})(\det H)^{\frac{1}{2}(T-m-n-1)}(\det\Sigma)^{\frac{1}{2}(T-m)}\mathrm{d}H$$

$$=\sum_{j=0}^{\infty}\frac{1}{j!}\sum_{\kappa}\frac{(\frac{1}{2}(T-m))_{\kappa}\Gamma_n(\frac{1}{2}(T-m))}{\Gamma(\frac{1}{2}n(T-m)+j)}C_{\kappa}(\tilde{W})\delta_j(\det\Sigma)^{\frac{1}{2}(T-m)} \quad (21)$$

where

$$\delta_j=\int_0^{\infty}g(z)z^{\frac{1}{2}n(T-m)+j-1}\mathrm{d}z$$

Thus, (19) can be written in the following form

$$\text{p.d.f}(e)=\frac{2^{-\frac{1}{2}n(T-m)}T^{\frac{1}{2}q}(2\pi)^{-\frac{1}{2}q}}{\left[\det(G(\partial W_a))\right]^{\frac{1}{2}}}\cdot g\{Te'G(\partial W_a)^{-1}e\}$$

$$\cdot\sum_{j=0}^{\infty}\frac{1}{j!}\sum_{\kappa}\frac{\left(\frac{T-m}{2}\right)_{\kappa}}{\Gamma\left(\frac{n(T-m)}{2}+j\right)}C_{\kappa}(\tilde{W})\delta_{j}\Big|_{W=0} \tag{22}$$

Hence, we have the following theorem

Theorem In the SUR model given by (2) and (4), we assume that U has the third kind elliptically contoured distribution, i. e. $U\sim LEC_{T\times(n+1)}(0,\Sigma\otimes I;g)$, and g is Taylor expansible on $(-\infty,+\infty)$. Then the probability density function of the estimate error $\alpha^{*}-\alpha$ can be expressed as the following form,

$$\mathrm{pdf}(e)=\frac{2^{-\frac{1}{2}n(T-m)}T^{\frac{1}{2}q}(2\pi)^{-\frac{1}{2}q}}{\left[\det(G(\partial W_{a}))\right]^{\frac{1}{2}}}g\{Te'G(\partial W_{a})^{-1}e\}$$

$$\cdot\sum_{j=0}^{\infty}\frac{1}{j!}\sum_{\kappa}\frac{\left(\frac{T-m}{2}\right)_{\kappa}}{\Gamma\left(\frac{1}{2}n(T-m)+j\right)}C_{\kappa}(\tilde{W})\delta_{j}\Big|_{W=0}$$

where $e=\alpha^{*}-\alpha$, $\tilde{W}=\Sigma^{-\frac{1}{2}}W\Sigma^{-\frac{1}{2}}$, $G(\partial W_{a})$ is an operator defined in the lemma of section 2, and δ_{j} is defined as in (21).

In the above theorem, if U is normal, i. e. $g(x)=\exp\left(-\frac{1}{2}x\right)$, through some evaluation, we can obtain the result of the exact distribution of $\alpha^{*}-\alpha$ as in Phillips[1].

References

[1] Phillips P C B, *Econometrica*, 53 (1985), 745～756

[2] Zellner A, *J. of American Statist. Association*, 1962, 58, 348～368

[3] Anderson T W et al., (1982): *On the theory of multivariate elliptically contoured distributions and their applications*, Technical Report No. 54, ONR Contract N00014-75-C-0442, Department of Statistics, Stanford Uni. Califonia

[4] Teng C Y et al., *J. of Math. Research & Exposition*, 9 (1989), 4, 64～73

[5] Muirhead R J , *Aspect of Multivariate Statistical Theory*, John Wiley & Sons, New York, 1982

SUR 模型参数估计量的精确分布

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摘 要

将指数函数的矩阵分式算子的结论，推广到无穷可微函数的情形．利用此结论，导出扰动项遵循第三类椭球等高分布的SUR模型估计量的精确分布，并推广了Phillips (1985) 的结果．

关键词 SUR模型，精确分布，分式积分算子，椭球等高分布

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