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边界拟合坐标系中的流动方程

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摘　要　本文提出了流体力学解析边界拟合坐标系方法，及建立在这种坐标系基础上对流动方程进行摄动处理的方法，并列举几种流动建立了解析边界拟合坐标系中的方程。

关键词　解析边界拟合坐标系，摄动法，流动方程

1　前　言

坐标系的选取，历来是流动问题求解中的一个重要技巧。所谓边界拟合坐标系，就是选取坐标系，使得它的坐标线与流动的边界相拟合。例如，两平行平板间的流动，取ox轴向为流动方向的直角坐标系，这就是一种边界拟合坐标系，在此坐标系中，两平行平板就是平行于ox轴的坐标线。

采用边界拟合坐标系，由于边界的几何描述变得简单，使得问题的边界条件的数学表述简化，对于许多流动问题，更主要的是使其投影形式的流动方程变得简单，能突出主流方向，便于采用尺度的比较，对方程进行逐级近似的摄动处理。

边界拟合坐标系形成，Thompson于1974年出于问题的数值求解形成高精度拟合求解区域的边界的计算网格，提出了一种采用数值方法形成边界拟合坐标系的方法[1]，在此我们不妨称其为数值边界拟合坐标系。严格地说，数值边界拟合坐标系是一种近似的边界拟合坐标系，其精度与数值求解椭圆型方程边值问题的精度有关。本文所提出的边界拟合坐标系是直接取流动的边界线(面)为坐标线(面)，将它们用几何参数解析地表示出来形成以边界线(面)为坐标线(面)的正交曲线坐标系，因此这是一种精确的边界拟合坐标，不妨称其为解析边界拟合坐标系。本文采用这种解析边界拟合坐标系，作为例子，给出几种不可压粘性流体流动的流动方程。

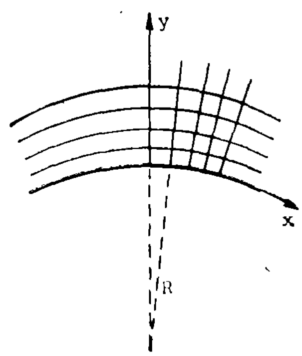

图1　曲面边界层流动
Fig.1　Boundary layer flow along a curved wall

2　曲面边界层流动[2]

考虑二维曲面边界的边界层，取坐标系：x轴沿曲面，y轴为曲面的法向（如图1所示）。这是一个正交曲线坐标系。坐标系中，任意两充分接近的点的距离为

$$(ds)^2=\left(\frac{R+y}{R}dx\right)^2+(dy)^2$$

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其中R为曲面的曲率半径，于是得到此曲线坐标系的拉梅系数

$$h_1=(R+y)/R \quad , \qquad h_2=1$$

因而在此曲线坐标系中，求得N—S方程（不可压粘性流体的流动方程）为：

$$\left\{\begin{aligned}
&\frac{\partial u}{\partial t}+\frac{R}{R+y}u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{uv}{R+y}=-\frac{R}{R+y}\frac{1}{\rho}\frac{\partial p}{\partial x}+\gamma\Big[\frac{R^2}{(R+y)^2}\frac{\partial^2 u}{\partial x^2}\\
&\quad+\frac{\partial^2 u}{\partial y^2}+\frac{1}{R+y}\frac{\partial u}{\partial y}-\frac{u}{(R+y)^2}+\frac{2R}{R+y}\frac{\partial v}{\partial x}-\frac{Rv}{(R+y)^3}\frac{\mathrm{d}R}{\mathrm{d}x}\\
&\quad+\frac{Ry}{(R+y)^3}\frac{\mathrm{d}R}{\mathrm{d}x}\frac{\partial u}{\partial x}\Big]\\
&\frac{\partial v}{\partial t}+\frac{R}{R+y}u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}-\frac{u^2}{R+y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\gamma\Big[\frac{R^2}{(R+y)^2}\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\\
&\quad+\frac{1}{R+y}\frac{\partial v}{\partial y}-\frac{v}{(R+y)^2}-\frac{2R}{(R+y)^2}\frac{\partial u}{\partial x}+\frac{Ry}{(R+y)^3}\frac{\mathrm{d}R}{\mathrm{d}x}\frac{\partial v}{\partial x}\\
&\quad+\frac{Ru}{(R+y)^3}\frac{\mathrm{d}R}{\mathrm{d}x}\Big]\\
&\frac{R}{R+y}\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{v}{R+y}=0
\end{aligned}\right.$$

设δ为边界层的厚度尺度，L为曲面的纵向（x轴向）尺度，U与V分别为x轴向和y轴向的速度分量尺度，则由边界层理论有

$$\frac{\delta}{L}=\varepsilon\ll 1 \quad , \qquad \frac{V}{U}=\varepsilon\ll L$$

且雷诺数 $\mathrm{Re}=\dfrac{UL}{\gamma}=\dfrac{1}{\varepsilon^2}$

考虑曲面的曲率半径与纵向尺度同量级的情形，即$R\sim L$。因此曲面的曲率$1/R=\varepsilon\ll 1$，为微弯曲面。在此情形下将上述N—S方程无量纲化。取变量

$$x=Lx' \quad , \qquad y=\varepsilon Ly' \quad , \qquad t=\frac{L}{U}t' \quad , \qquad u=Uu' \quad ,$$

$$v=\varepsilon Uv' \quad , \qquad p=\rho U^2p' \quad , \qquad R=LR'$$

则N—S方程无量纲化为

$$\left\{\begin{aligned}
&\frac{\partial u}{\partial t}+\frac{R}{R+\varepsilon y}u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\varepsilon}{R+\varepsilon y}uv=-\frac{R}{R+\varepsilon y}\frac{\partial p}{\partial x}+\frac{R^2\varepsilon^2}{(R+\varepsilon y)^2}\frac{\partial^2 u}{\partial x^2}\\
&\quad+\frac{\partial^2 u}{\partial y^2}+\frac{\varepsilon}{R+\varepsilon y}\frac{\partial u}{\partial y}-\frac{\varepsilon^2 u}{(R+\varepsilon y)^2}+\frac{2R\varepsilon^3}{(R+\varepsilon y)^2}\frac{\partial v}{\partial x}-\frac{R\varepsilon^3}{(R+\varepsilon y)^3}\frac{\mathrm{d}R}{\mathrm{d}x}\\
&\quad+\frac{Ry\varepsilon^3}{(R+\varepsilon y)^3}\frac{\mathrm{d}R}{\mathrm{d}x}\frac{\partial u}{\partial x}\\
&\varepsilon\frac{\partial v}{\partial t}+\frac{R\varepsilon}{R+\varepsilon y}u\frac{\partial v}{\partial x}+\varepsilon v\frac{\partial v}{\partial y}-\frac{u^2}{R+\varepsilon y}=-\frac{1}{\varepsilon}\frac{\partial p}{\partial y}+\frac{R\varepsilon^3}{(R+\varepsilon y)^2}\frac{\partial^2 v}{\partial x^2}\\
&\quad+\varepsilon\frac{\partial^2 v}{\partial y^2}+\frac{\varepsilon^2}{R+\varepsilon y}\frac{\partial v}{\partial y}-\frac{\varepsilon^2 v}{(R+\varepsilon y)^2}-\frac{2R\varepsilon^2}{(R+\varepsilon y)^2}\frac{\partial u}{\partial x}\\
&\quad+\frac{Ry\varepsilon^4}{(R+\varepsilon y)^3}\frac{\partial v}{\partial x}-\frac{\varepsilon^2}{(R+\varepsilon y)^3}u\frac{\mathrm{d}R}{\mathrm{d}x}\\
&\frac{R}{R+\varepsilon y}\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\varepsilon}{R+\varepsilon y}v=0
\end{aligned}\right.$$

注意到，为简便起见，我们将无量纲变量记号上的一撇省略了。

若采取逐次近似的方法求解，令

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots$$

$$v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots$$

$$p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \cdots$$

则得逐次近似方程

0 级近似

$$\begin{cases} \dfrac{\partial u_0}{\partial t} + u_0 \dfrac{\partial u_0}{\partial x} + v_0 \dfrac{\partial v_0}{\partial y} = -\dfrac{\partial p_0}{\partial x} + \dfrac{\partial^2 u_0}{\partial y^2} \\ \dfrac{\partial p_0}{\partial y} = 0 \\ \dfrac{\partial u_0}{\partial x} + \dfrac{\partial v_0}{\partial y} = 0 \end{cases}$$

0 级近似没有考虑平面弯曲的影响，实际为平板边界层。这里得到的方程也正好是熟知的平板边界层的方程。

1 级近似

$$\begin{cases} \dfrac{\partial u_1}{\partial t} + u_0 \dfrac{\partial u_1}{\partial x} + v_0 \dfrac{\partial u_1}{\partial y} + u_1 \dfrac{\partial u_0}{\partial x} + v_1 \dfrac{\partial u_0}{\partial y} = -\dfrac{\partial p_1}{\partial x} + \dfrac{\partial^2 u_1}{\partial y^2} \\ \qquad + \left[k \dfrac{\partial p_0}{\partial x} - \dfrac{1}{k} \dfrac{\partial u_0}{\partial y} + k u_0 \dfrac{\partial u_0}{\partial x} - k u_0 v_0 \right] \\ \dfrac{u_0^2}{R} = \dfrac{\partial p_1}{\partial y} \\ \dfrac{\partial u_1}{\partial x} + \dfrac{\partial v_1}{\partial y} = k \dfrac{\partial u_0}{\partial x} - \dfrac{v_0}{R} \end{cases}$$

其中， $k = \dfrac{y}{R}$

1 级近似考虑了曲率的影响，在没有考虑曲率影响的平板边界层中，在边界层内压力p在厚度方向(y轴向)是不发生变化的，可考虑平板弯曲后，在厚度方向就有压力的变化。

2 级近似

$$\begin{cases} \dfrac{\partial u_2}{\partial t} + u_0 \dfrac{\partial u_2}{\partial x} + v_0 \dfrac{\partial u_2}{\partial y} + u_2 \dfrac{\partial u_0}{\partial x} + v_2 \dfrac{\partial u_0}{\partial y} = -\dfrac{\partial p}{\partial x} + \dfrac{\partial^2 u_2}{\partial y^2} \\ \qquad + \left[k \dfrac{\partial p_1}{\partial x} - k^2 \dfrac{\partial p_0}{\partial x} + \dfrac{\partial^2 u_0}{\partial x^2} + \dfrac{1}{k} \dfrac{\partial u_1}{\partial y} - \dfrac{1}{k} \dfrac{\partial u_0}{\partial y} - k^2 u_0 \dfrac{\partial u_0}{\partial x} - k u_0 \dfrac{\partial u_1}{\partial x} \right. \\ \qquad \left. - k u_1 \dfrac{\partial u_0}{\partial x} - u_1 \dfrac{\partial u_1}{\partial x} - v_1 \dfrac{\partial u_1}{\partial y} - k u_0 v_1 - k u_1 v_0 - k u_0 v_0 \right] \\ \dfrac{\partial v_0}{\partial t} + u_0 \dfrac{\partial v_0}{\partial x} + v_0 \dfrac{\partial v_0}{\partial y} - \dfrac{1}{k}(u_0 u_1 - k u_0^2) - \dfrac{\partial^2 v_0}{\partial y^2} = -\dfrac{\partial p_2}{\partial y} \\ \dfrac{\partial u_2}{\partial x} + \dfrac{\partial v_2}{\partial y} = k \dfrac{\partial u_1}{\partial x} - k^2 \dfrac{\partial u_0}{\partial x} - \dfrac{v_1}{k} + \dfrac{v_0}{k} = 0 \end{cases}$$

3　弯曲平面渠道流动

取坐标系：x轴与渠道壁拟合，指向流动方向；y轴垂直渠道壁(如图2所示)。在坐标系中微元长为

$$(\mathrm{d}s)^2=\left(\frac{R+y}{R}\mathrm{d}x\right)^2+(\mathrm{d}y)^2$$

$$h_1=\frac{R+y}{R}\qquad h_2=1$$

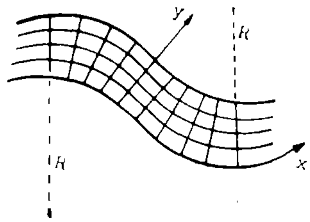

图2　弯曲渠道流动
Fig.2　Flow in a curved channel

流动沿渠道，只有沿渠道的分量u，垂直于渠道的速度分量 $v\equiv0$。因此在此坐标系中，不可压粘性流的流动方程为

$$\begin{cases}\dfrac{\partial u}{\partial t}=-\dfrac{R}{R+y}\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}+\gamma\left[\dfrac{\partial^2 u}{\partial y^2}+\dfrac{1}{R+y}\dfrac{\partial u}{\partial y}-\dfrac{u}{(R+y)^2}\right]\\ -\dfrac{u^2}{R+y}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial y}+\gamma\dfrac{Ru}{(R+y)^3}\dfrac{\mathrm{d}R}{\mathrm{d}x}\\ \dfrac{\partial u}{\partial x}=0\end{cases}$$

无量纲化，取变量为

$$t=Tt',\quad u=Uu',\quad x=TUx',\quad y=\varepsilon TUx',\quad R=TUR'$$

得到

$$\begin{cases}\dfrac{\partial u}{\partial t}=-\dfrac{R}{R+\varepsilon y}\dfrac{\partial p}{\partial x}+\dfrac{1}{\mathrm{Re}}\left[\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\varepsilon}{R+\varepsilon y}\dfrac{\partial u}{\partial y}-\dfrac{\varepsilon^2}{(R+\varepsilon y)^2}u\right]\\ -\dfrac{\varepsilon}{R+\varepsilon y}u^2=-\dfrac{\partial p}{\partial y}+\dfrac{1}{\mathrm{Re}}\dfrac{R\varepsilon}{(R+\varepsilon y)^3}\dfrac{\mathrm{d}R}{\mathrm{d}x}\\ \dfrac{\partial u}{\partial x}=0\end{cases}$$

对微弯的渠道，渠道曲率很小，即 $\varepsilon\ll1$。渠宽尺度与渠道曲率半径之比是一个小量。

令　$u=u_0+\varepsilon u_1+\varepsilon^2u_2+\cdots$

$p=p_0+\varepsilon p_1+\varepsilon^2p_2+\cdots$

流动方程可写为如下逐级近似的形式。

0级近似

$$\begin{cases}\dfrac{\partial u_0}{\partial t}=-\dfrac{\partial p_0}{\partial x}+\dfrac{1}{\mathrm{Re}}\dfrac{\partial^2 u_0}{\partial y^2}\\ \dfrac{\partial p_0}{\partial y}=0\\ \dfrac{\partial u_0}{\partial x}=0\end{cases}$$

0 级近似没有渠道弯曲的影响，因此就是两平行平板间的平面平行流动。

1 级近似

$$\begin{cases}\dfrac{\partial u_1}{\partial t}=-\dfrac{\partial p}{\partial x}+k\dfrac{\partial p_0}{\partial x}+\dfrac{1}{\mathrm{Re}}\left(\dfrac{\partial^2 u_1}{\partial y^2}+\dfrac{k}{R}\dfrac{\partial u_0}{\partial y}\right)\\ -\dfrac{1}{R}u_0^2=-\dfrac{\partial p_1}{\partial y}+\dfrac{1}{R^2}\dfrac{1}{\mathrm{Re}}\dfrac{\mathrm{d}R}{\mathrm{d}x}u_0\\ \dfrac{\partial u_1}{\partial x}=0\end{cases}$$

2 级近似

$$\begin{cases}\dfrac{\partial u_2}{\partial t}=-\dfrac{\partial p_2}{\partial x}+k\dfrac{\partial p_1}{\partial x}-k^2\dfrac{\partial p_0}{\partial x}+\dfrac{1}{\mathrm{Re}}\left[\dfrac{\partial^2 u_2}{\partial y^2}+\dfrac{1}{R}\dfrac{\partial u_1}{\partial y}-\dfrac{k}{R}\dfrac{\partial u_0}{\partial y}-\dfrac{u}{R^2}\right]\\ -\dfrac{1}{R}(u_0u_1-ku_0^2)=-\dfrac{\partial p^2}{\partial y}-3\dfrac{1}{\mathrm{Re}}\dfrac{k}{\mathrm{R}^2}\dfrac{\mathrm{d}R}{\mathrm{d}x}u_0+\dfrac{1}{\mathrm{Re}}\dfrac{1}{R^2}\dfrac{\mathrm{d}R}{\mathrm{d}x}u_1\\ \dfrac{\partial u_2}{\partial x}=0\end{cases}$$

4　Teffery-Hamel流[3]

人们将不可压粘性流体在成一角度的两平板间的收缩或扩散的流动称为Jeffery-Hamel流，文〔3〕得到了用椭圆函数表示的流动的精确解，其求解所使用的是极坐标。对于这种流动来说，极坐标也是一种边界拟合坐标系，因为两平板边界在极坐标系中正好是坐标线。现在引进另一种边界拟合坐标系，这对于小角度的Jeffery-Hamel流来说，显然是方便于逐级逼近求解的。

如图 3 所示，取坐标系：ox轴为径向（如同极坐标系），oy轴为以两平板的交点为圆心，某一半径所划的圆弧。在此坐标系中，微元长为

$$(\mathrm{d}s)^2=(\mathrm{d}x)^2+\left(\frac{R+x}{R}\mathrm{d}y\right)^2$$

于是拉梅系数　$h_1=1$，　$h_2=\dfrac{R+x}{R}$

流动只在ox轴向有分量，设为u，因此不可压粘性流体层流流动的方程为

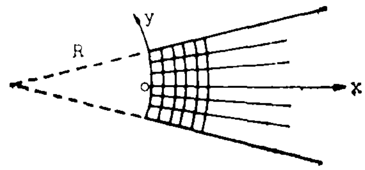

图 3　Jeffery-Hamel流

Fig.3　Jeffery-Hamel flow

$$\frac{\partial u}{\partial x}+\frac{u}{R+x}=0$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\gamma\left[\frac{\partial^2 u}{\partial x^2}+\frac{R^2}{(R+x)^2}\frac{\partial^2 u}{\partial y^2}+\frac{1}{R+x}\frac{\partial u}{\partial x}-\frac{u}{(R+x)^2}\right]$$

无量纲化，取变量

$$x=Lx',\ y=Ly',\ u=\frac{L}{T}u',\quad p=\frac{\rho L^2}{T^2}p',\quad R=\frac{L}{\varepsilon}R',\quad \mathrm{Re}=\frac{L^2}{\gamma T},$$

$$\varepsilon=\frac{L}{R}=\theta\ (\text{平板夹角，弧度})$$

则流动方程变为

$$\begin{cases}\dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}=-\dfrac{\partial p}{\partial y}+\dfrac{1}{\mathrm{Re}}\left[\dfrac{\partial^2 u}{\partial x^2}+\dfrac{1}{(1+\varepsilon x)^2}\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\varepsilon}{1+\varepsilon x}\dfrac{\partial u}{\partial x}-\dfrac{\varepsilon^2 u}{(1+\varepsilon x)^2}\right]\\ 0=-\dfrac{1}{1+\varepsilon x}\dfrac{\partial p}{\partial y}+\dfrac{1}{\mathrm{Re}}\dfrac{\varepsilon}{R(H\varepsilon x)^2}\dfrac{\partial u}{\partial y}\\ \dfrac{\partial u}{\partial x}+\dfrac{u}{1+\varepsilon x}=0\end{cases}$$

对于小夹角情况，写为逐级近似形式

0 级近似

$$\begin{cases}\dfrac{\partial u_0}{\partial t}=-\dfrac{\partial p_0}{\partial x}+\dfrac{1}{\mathrm{Re}}\dfrac{\partial^2 u_0}{\partial y^2}\\ \dfrac{\partial u_0}{\partial x}=0\end{cases}$$

这就是平行平板间的流动方程.

1 级近似

$$\begin{cases}\dfrac{\partial u_1}{\partial t}+u_0\dfrac{\partial u_1}{\partial x}=-\dfrac{\partial p_1}{\partial x}+\dfrac{1}{\mathrm{Re}}\left(\dfrac{\partial^2 u_1}{\partial x^2}+\dfrac{\partial^2 u_1}{\partial y^2}-2x\dfrac{\partial^2 u_0}{\partial y^2}\right)\\ 0=-\dfrac{\partial p_1}{\partial y}+\dfrac{1}{\mathrm{Re}}\dfrac{1}{R}\dfrac{\partial u_0}{\partial y}\\ \dfrac{\partial u_1}{\partial x}+u_0=0\end{cases}$$

2 级近似

$$\dfrac{\partial u_2}{\partial t}+u_0\dfrac{\partial u_2}{\partial x}+u_1\dfrac{\partial u_1}{\partial x}=-\dfrac{\partial p_2}{\partial x}+\dfrac{1}{R_e}\left(\dfrac{\partial^2 u_2}{\partial x^2}+\dfrac{\partial u_1}{\partial x}-u_0+\dfrac{\partial^2 u_2}{\partial y^2}-2x\dfrac{\partial^2 u_1}{\partial y^2}\right)+3x^2\dfrac{\partial^2 u_0}{\partial y^2}$$

$$0=-\dfrac{\partial p_2}{\partial y}+x\dfrac{\partial p_1}{\partial y}+\dfrac{1}{\mathrm{Re}}\dfrac{1}{R}\left[\dfrac{\partial u_1}{\partial y}-2x\dfrac{\partial u_0}{\partial y}\right]$$

$$\dfrac{\partial u_2}{\partial x}+u_2-xu_0=0$$

5 弯曲圆管流动

假设圆管的弯曲是在平面内的弯曲. 取正交坐标系(如图4所示): x与圆管中心线一致，为平面内的曲线; r在圆截面内;径向; φ在圆截面内,圆周向. 由于在坐标系中微元

$$(ds)^2=\left(\frac{R+r\cos\varphi}{R}dx\right)^2+(dr)^2+(rd\varphi)^2$$

$$\therefore\quad h_1=\frac{R+r\cos\varphi}{R},\qquad h_2=1,\qquad h_3=r$$

流动只有轴向分量u，而$v\equiv0$，$w\equiv0$.因此在此坐标

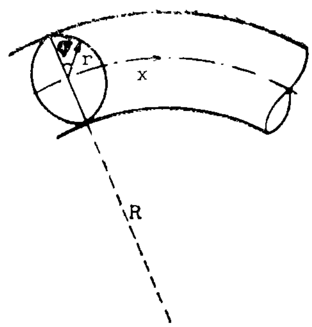

图4 弯曲圆管流动

Fig.4 Flow in a curved pipe

系中，不可压缩粘性流体流动方程为

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = -\dfrac{R}{R+r\cos\varphi}\dfrac{1}{\rho}\dfrac{\mathrm{d}p}{\mathrm{d}x} + \gamma\dfrac{\partial^2 u}{\partial r^2} + \dfrac{\gamma}{R+r\cos\varphi}\left(\dfrac{R}{r}+2\cos\varphi\right)\dfrac{\partial u}{\partial r} \\
\qquad + \dfrac{\gamma}{r^2}\dfrac{\partial^2 u}{\partial\varphi^2} - \dfrac{\sin\varphi}{R+r\cos\varphi}\dfrac{1}{r}\dfrac{\partial u}{\partial\varphi} - \dfrac{1}{(R+r\cos\varphi)^2}u \\
\dfrac{u^2\cos\varphi}{R+r\cos\varphi} = \dfrac{1}{\rho}\dfrac{\partial p}{\partial r} - \dfrac{\gamma R\cos\varphi}{(R+r\cos\varphi)^3}\dfrac{\mathrm{d}R}{\mathrm{d}x}u \\
\dfrac{u^2\sin\varphi}{R+r\cos\varphi} = -\dfrac{1}{\rho}\dfrac{1}{r}\dfrac{\partial p}{\partial\varphi} - \dfrac{\gamma R\sin\varphi}{(R+r\cos\varphi)^3}\dfrac{\mathrm{d}R}{\mathrm{d}x}u \\
\dfrac{\partial u}{\partial x} = 0
\end{cases}
$$

设　　$\dfrac{r_0}{R_0} = \varepsilon$（$r_0$为圆管半径，$R_0$为圆管特征曲率）

无量纲化　$t = Tt'$，$r = r_0 r'$，$R = \dfrac{r_0}{\varepsilon}R'$，　$u = \dfrac{r_0}{T}u'$，　$p = \dfrac{r_0^2}{T^2}p'$，　$\mathrm{Re} = \dfrac{r_0^2}{\Upsilon T}$

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = -\dfrac{R}{R+\varepsilon r\cos\varphi}\dfrac{\partial p}{\partial x} + \dfrac{1}{\mathrm{Re}}\left\{\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{R+\varepsilon r\cos\varphi}\left(\dfrac{R}{r}+2\varepsilon\cos\varphi\right)\dfrac{\partial u}{\partial r}\right. \\
\qquad \left. + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial\varphi^2} - \dfrac{\varepsilon\sin\varphi}{R+\varepsilon r\cos\varphi}\dfrac{1}{r}\dfrac{\partial u}{\partial\varphi} - \dfrac{\varepsilon^2}{(R+\varepsilon r\cos\varphi)^2}u\right\} \\
\dfrac{\varepsilon\cos\varphi}{R+\varepsilon r\cos\varphi}u^2 = \dfrac{\partial p}{\partial r} - \dfrac{\varepsilon R\cos\varphi}{(R+\varepsilon r\cos\varphi)^3}\dfrac{1}{\mathrm{Re}}\dfrac{\mathrm{d}R}{\mathrm{d}x}u \\
\dfrac{\varepsilon\sin\varphi}{R+\varepsilon r\cos\varphi}u^2 = -\dfrac{1}{r}\dfrac{\partial p}{\partial\varphi} - \dfrac{1}{\mathrm{Re}}\dfrac{\varepsilon R\sin\varphi}{(R+\varepsilon r\cos\varphi)^3}\dfrac{\mathrm{d}R}{\mathrm{d}x}u
\end{cases}
$$

若管道微弯　　$\varepsilon \ll 1$.　则有逐级近似方程

0 级近似

$$
\begin{cases}
\dfrac{\partial u_0}{\partial t} = -\dfrac{\partial p_0}{\partial x} + \dfrac{1}{\mathrm{Re}}\left(\dfrac{\partial^2 u_0}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u_0}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u_0}{\partial\varphi^2}\right) \\
0 = \dfrac{\partial p_0}{\partial r} \\
0 = \dfrac{\partial p}{\partial\varphi}
\end{cases}
$$

不考虑圆管弯曲的影响，刚好就是直圆管的流动方程。

1 级近似

$$
\begin{cases}
\dfrac{\partial u_1}{\partial t} = -\dfrac{\partial p_1}{\partial x} + \dfrac{1}{\mathrm{Re}}\left(\dfrac{\partial^2 u_1}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u_1}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u_1}{\partial\varphi^2}\right) + k\dfrac{\partial p_0}{\partial x} \\
\qquad + \dfrac{1}{\mathrm{Re}}\left(-\dfrac{k}{r}\dfrac{\partial u_0}{\partial r} + \dfrac{2\cos\varphi}{R}\dfrac{\partial u_0}{\partial r} - \dfrac{\sin\varphi}{R}\dfrac{1}{r}\dfrac{\partial u_0}{\partial\varphi}\right) \\
\dfrac{\cos\varphi}{R}u_0^2 = \dfrac{\partial p_1}{\partial r} - \dfrac{1}{\mathrm{Re}}\dfrac{\cos\varphi}{R^2}\dfrac{\mathrm{d}R}{\mathrm{d}x}u_0 \\
\dfrac{\sin\varphi}{R}u_0^2 = -\dfrac{1}{r}\dfrac{\partial p}{\partial\varphi} - \dfrac{1}{\mathrm{Re}}\dfrac{\sin\varphi}{R^2}\dfrac{\mathrm{d}R}{\mathrm{d}x}u_0
\end{cases}
$$

其中 $k=\frac{r}{R}\cos\varphi$

2 级近似

$$\begin{cases}\frac{\partial u_2}{\partial t}=-\frac{\partial p_2}{\partial x}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u_2}{\partial r^2}+\frac{1}{r}\frac{\partial u_2}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_2}{\partial\varphi^2}\right)+k\frac{\partial p_1}{\partial x}-k^2\frac{\partial p_0}{\partial x}\\ \quad+\frac{1}{\mathrm{Re}}\left(-\frac{k}{r}\frac{\partial u_1}{\partial r}+\frac{2\cos\varphi}{R}\frac{\partial u_1}{\partial r}-\frac{\sin\varphi}{R}\frac{1}{r}\frac{\partial u_1}{\partial\varphi}\right)+\frac{1}{\mathrm{Re}}\left(\frac{k^2}{r}\frac{\partial u_0}{\partial r}\right.\\ \quad\left.-\frac{2k\cos\varphi}{R}\frac{\partial u_0}{\partial r}+\frac{r\cos\varphi}{R^2}\frac{1}{r}\frac{\partial u_0}{\partial r}+\frac{k\sin\varphi}{R}\frac{1}{r}\frac{\partial u_0}{\partial\varphi}-\frac{u_0}{R^2}\right)\\ \frac{2\cos\varphi}{R}u_0u_1-\frac{k\cos\varphi}{R}u_0^2=\frac{\partial p_2}{\partial r}-\frac{1}{\mathrm{Re}}\frac{\cos\varphi}{R_2}\frac{\mathrm{d}R}{\mathrm{d}x}u_1+\frac{1}{\mathrm{Re}}\frac{3k\cos\varphi}{R^2}\frac{\mathrm{d}R}{\mathrm{d}x}u_0\\ \frac{2\sin\varphi}{R}u_0u_1-\frac{k\sin\varphi}{R}u_0^2=-\frac{1}{r}\frac{\partial p_2}{\partial\varphi}-\frac{1}{\mathrm{Re}}\frac{\sin\varphi}{R^2}\frac{\mathrm{d}R}{\mathrm{d}x}u_1+\frac{1}{\mathrm{Re}}\frac{3k\sin\varphi}{R^2}\frac{\mathrm{d}R}{\mathrm{d}x}u_0\end{cases}$$

6 收缩圆管流动

取坐标系(如图 5 所示)：x为管道轴向；r为管道以R为半径的圆弧；φ为管道绕管轴的迴转角。坐标原点是管轴和以R 为半径、管道收缩中心为心的球面的交点，则在坐标系中微元长

$$(\mathrm{d}s)^2=(\mathrm{d}x)^2+\left(\frac{R+x}{R}\mathrm{d}r\right)^2+\left[\frac{(R+x)r}{R}\mathrm{d}\varphi\right]^2$$

于是拉梅系数

$$h_1=1\ ,\quad h_2=\frac{R+x}{R},\quad h_3=\frac{(R+x)r}{R}$$

层流流动沿x轴向，故只在ox轴向有分量u，在 r 方向和φ方向都没有流动。即$v\equiv w\equiv 0$。

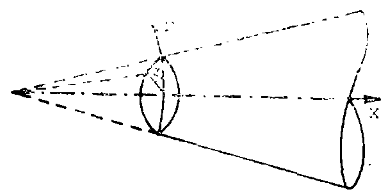

图 5 收缩（或扩散）圆管流动
Fig.5 Flow in a convergent (or divergent)pipe

于是得到不可压粘性流体流动在此坐标系中的流动方程

$$\begin{cases}\frac{\partial u}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\gamma\frac{\partial^2 u}{\partial r^2}+\frac{\gamma R^2}{(R+x)^2 r}\frac{\partial u}{\partial r}+\frac{2\gamma}{R+x}\frac{\partial u}{\partial x}-\frac{4\gamma u}{(R+x)^2}\\ 0=-\frac{R}{R+x}\frac{1}{\rho}\frac{\partial p}{\partial r}+\frac{2\gamma R}{(R+x)^2}\frac{\partial u}{\partial r}\\ 0=\frac{\partial p}{\partial\varphi}\\ \frac{\partial u}{\partial x}+\frac{2u}{R+x}=0\end{cases}$$

无量纲化，设 $\frac{r_0}{R}=\varepsilon$ （r_0 为管道相应于R的截面处的半径） 变量

$$t=Tt',\ x=r_0x',\ r=r_0r',\ u=\frac{r_0}{T}u',\quad p=\frac{pr_0^2}{T^2}p',\quad R=\frac{r_0}{\varepsilon}R',\mathrm{Re}=\frac{r_0^2}{\gamma T}$$

则流动方程化为如下无量纲形式

$$\begin{cases}\dfrac{\partial u}{\partial t}=-\dfrac{\partial p}{\partial x}+\dfrac{1}{\mathrm{Re}}\left\{\dfrac{\partial^2 u}{\partial r^2}+\dfrac{R^2}{(R+\varepsilon x)^2}\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{2\varepsilon}{R+\varepsilon x}\dfrac{\partial u}{\partial x}-\dfrac{4\varepsilon^2}{(R+\varepsilon x)^2}u\right.\\ 0=-\dfrac{R}{R+\varepsilon x}\dfrac{\partial p}{\partial r}+\dfrac{1}{\mathrm{Re}}\dfrac{2R\varepsilon}{(R+\varepsilon x)^2}\dfrac{\partial u}{\partial r}\\ \dfrac{\partial u}{\partial x}+\dfrac{2\varepsilon}{R+\varepsilon x}u=0\end{cases}$$

对于微收缩管道　$\varepsilon\ll 1$，可将方程写为逐级近似的形式：

0 级近似

$$\begin{cases}\dfrac{\partial u_0}{\partial t}=-\dfrac{\partial p_0}{\partial x}+\dfrac{1}{\mathrm{Re}}\left(\dfrac{\partial^2 u_0}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u_0}{\partial r}\right)\\ 0=\dfrac{\partial p_0}{\partial r}\\ \dfrac{\partial u_0}{\partial x}=0\end{cases}$$

0 级近似，不计管道的收缩影响，所以正好是直圆管的流动方程。

1 级近似

$$\begin{cases}\dfrac{\partial u_1}{\partial t}=-\dfrac{\partial p_1}{\partial x}+\dfrac{1}{\mathrm{Re}}\left(\dfrac{\partial^2 u_1}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u_1}{\partial r}-\dfrac{2k}{r}\dfrac{\partial u_0}{\partial r}+\dfrac{2}{R}\dfrac{\partial u_0}{\partial x}\right)\\ -\dfrac{\partial p_1}{\partial r}=-k\dfrac{\partial p_0}{\partial r}+\dfrac{1}{\mathrm{Re}}\dfrac{2}{R}\dfrac{2u_0}{\partial r}\\ \dfrac{\partial u_1}{\partial x}+\dfrac{2}{R}u_0=0\end{cases}$$

其中 $k=\dfrac{r}{R}$

2 级近似

$$\begin{cases}\dfrac{\partial u_2}{\partial t}=-\dfrac{\partial p_2}{\partial x}+\dfrac{1}{\mathrm{Re}}\left(\dfrac{\partial^2 u_2}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u_2}{\partial r}-\dfrac{2k}{r}\dfrac{\partial u_1}{\partial r}+\dfrac{3k^2}{r}\dfrac{\partial u_0}{\partial r}+\dfrac{2}{R}\dfrac{\partial u_1}{\partial x}-\dfrac{2k}{R}\dfrac{\partial u_0}{\partial x}-\dfrac{4}{R^2}u_0\right.\\ -\dfrac{\partial p_2}{\partial r}=-k\dfrac{\partial p_1}{\partial r}+k^2\dfrac{\partial p_0}{\partial r}+\dfrac{1}{\mathrm{Re}}\left(\dfrac{2}{R}\dfrac{\partial u_1}{\partial r}-\dfrac{2k}{R}\dfrac{\partial u_0}{\partial r}\right)\\ \dfrac{\partial u_2}{\partial x}+\dfrac{2}{R}u_1-\dfrac{2k}{R}u_0=0\end{cases}$$

7　结束语

从本文列举的几则例子可以看出，由于采用了边界拟合坐标系，使坐标系的选取适合于流动的特征，从而使描述流动的方程得到了简化。结果表明了边界拟合坐标系的优势以及如何根据流动的特征来建立其边界拟合坐标系。就如何根据流动特征建立边界拟合坐标系而言，本文所讨论的几种流动是具有代表性和典型性的，其建立的边界拟合坐标系及建立边界拟合坐标系的方法可以推广应用。例如：①弯曲渠道流、弯曲管道流和收缩流的边界拟合坐标系，可推广应用于相应问题的流动稳定性问题，研究弯曲或收缩

对流动稳定性的影响，还可推广应用于这些问题的非牛顿流体流动；②对于弯曲渠道流的边界拟合坐标系，可推广应用到弯曲渠道具有自由表面的流动，考虑在弯曲渠道中自由表面波的运动等。值得指出的是，这种解析的边界拟合坐标系也可以应用到数值模拟中去，与Thompson的方法比较，虽有一定的局限性，但具有求解精度高，大大节省计算量的优点，对用Thompson方法形成边界拟合坐标系还可起指导的作用。

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The Flow Equations in Boundary Fitted-Coordinate System

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Abstract This paper proposed analytical boundars-fitted coordinate system in fluid mechanics and established the motion equations in this coordinate system for several kinds of flow.

Keywords analytical boundary fitted-coordinate, perturbation method, flow equations

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