

# 一类拟线性退缩椭圆组弱解的全局正则性

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**摘要** 利用 Muckenhoupt  $A_2$  类权函数的 Sobolev 空间  $H^1(\Omega, \mathbb{R}^n, \lambda)$  和  $H_0^1(\Omega, \mathbb{R}, \lambda)$  嵌入不等式, 证明一类拟线性退缩椭圆组弱解的  $L^p$  估计和全局正则性.

**关键词** 退缩椭圆组,  $L^p$  估计, 全局正则性

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文[1]建立了带 Muckenhoupt<sup>[2]</sup>权函数的 Sobolev 嵌入不等式和 Poincaré 不等式. 文[3]建立了一类非线性退缩椭圆组弱解梯度的  $L^p$  估计. 文[4]证明了对角形退缩椭圆组弱解在某一特殊结构下的局部正则性. 本文证明一类拟线性退缩椭圆组弱解的全局正则性, 把 Giaquinta<sup>[5]</sup>对同类形椭圆组弱解的局部正则性推广到退缩情形弱解的全局正则性, 而且本文的增长条件还包含控制增长条件和低于二次的自然增长条件.

记号:  $\Omega \subset \mathbb{R}^n (n \geq 3)$ ,  $\partial\Omega$  表示  $\Omega$  的边界,  $\omega_n$  表示单位球体积,  $\lambda(x) \in A_2$ .

$$\lambda^k(E) = \int_E \lambda^k(x) dx, \quad k = \pm 1, \pm 2, \quad E \subseteq \Omega$$

$$u_R = \frac{1}{\lambda(B_R)} \int_{B_R} u(x) \lambda(x) dx, \quad \int_E u \lambda dx = \frac{1}{|E|} \int_E u(x) \lambda(x) dx$$

$$\int_{B_R} u \lambda dx = \lambda^{-1}(B_R) \int_{B_R} u(x) \lambda(x) dx$$

$$B_R^+ = B_R^+(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < R, x^n > 0\}$$

$$u_\rho^+ = \int_{B_\rho^+} u dx, \quad \Omega_R(x_0) = B_R(x_0) \cap \Omega$$

$$\Gamma_R = \{x \in \mathbb{R}^n : |x| < R, x^n = 0\}, \quad \Gamma_R(x_0) = \partial B_R^+ \setminus \partial B_R$$

$$\int_{B_R^+} u \lambda dx = \lambda^{-1}(B_R) \int_{B_R^+} u(x) \lambda(x) \setminus \mathcal{O} dx$$

$$\phi(x_0, \rho) = \int_{B_\rho^+(x_0)} |\nabla u|^2 dx, \quad \phi(0, \rho) = \phi(\rho)$$

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## 1 加权 Sobolev-Poincaré 不等式

记  $r$  为  $H^1(\Omega, \lambda)$  的临界嵌入指数, 由 [3] 有下列引理

**引理 1** 设  $\lambda(x) \in A_2$ , 且有界. 则存在正数  $R_0, C_1$  和  $C_2$  使得

$$C_1 R^{\frac{2r}{r-2}} \leq \lambda(B_R) \leq C_2 R^n, \forall R \leq R_0, \frac{2n}{n-1} < r \leq \frac{2n}{n-2} \quad (1)$$

**引理 2** 设  $\lambda(x) \in A_p, 1 < p < \infty, u \in H_0^{1,p}(\Omega, \mathbb{R}^N, \lambda)$ . 则对  $\forall B_R \subset \Omega, \forall k \in [1, r/p]$  有

$$\left(\frac{1}{\lambda(B_R)} \int_{B_R} |u|^{kp} \lambda dx\right)^{\frac{1}{kp}} \leq CR \left(\frac{1}{\lambda(B_R)} \int_{B_R} |\nabla u|^p dx\right)^{1/p} \quad (2)$$

**引理 3** 设  $\lambda(x) \in A_p, 1 < p < \infty, u \in H^{1,p}(\Omega, \mathbb{R}^N, \lambda)$ . 则对  $\forall B_R \subset \Omega, \forall k \in [1, r/p]$  有

$$\left(\frac{1}{\lambda(B_R)} \int_{B_R} |u - u_R|^{kp} \lambda dx\right)^{\frac{1}{kp}} \leq CR \left(\frac{1}{\lambda(B_R)} \int_{B_R} |\nabla u|^p \lambda dx\right)^{\frac{1}{p}} \quad (3)$$

利用引理 1 可得加权 Sobolev-Poincaré 不等式:

$$\left(\int_{B_R} |u|^r \lambda dx\right)^{\frac{1}{r}} \leq C \left(\int_{B_R} |\nabla u|^2 \lambda dx\right)^{\frac{1}{2}}, \forall u \in H_0^1(B_R, \mathbb{R}^N, \lambda) \quad (4)$$

$$\left(\int_{B_R} |u - u_R|^r \lambda dx\right)^{\frac{1}{r}} \leq C \left(\int_{B_R} |\nabla u|^2 \lambda dx\right)^{\frac{1}{2}}, \forall u \in H^1(B_R, \mathbb{R}^N, \lambda) \quad (5)$$

**引理 4** 设  $\lambda(x) \in A_p, 1 < p < \infty$ , 若存在  $\alpha > 0, x_0 \in \bar{\Omega}, \Omega_\rho(x_0) \subset \Omega$ , 使得  $|B_\rho \setminus \Omega| \geq \alpha |B_\rho|$ . 则对于  $u \in C^1(\bar{\Omega}_\rho(x_0))$ ,  $u$  在  $\partial\Omega \cap B_\rho$  上为零,  $\forall k \in [1, r/p]$  有

$$\left(\frac{1}{\lambda(B_\rho)} \int_{\Omega_\rho(x_0)} |u(x)|^{kp} \lambda dx\right)^{\frac{1}{kp}} \leq C \rho \left(\frac{1}{\lambda(B_\rho)} \int_{\Omega_\rho(x_0)} |\nabla u|^p \lambda dx\right)^{\frac{1}{p}} \quad (6)$$

引理 4 的证明与 [1] 中定理 1.5 和引理 1.2 相同.

**附注** 由于  $C^\infty(\bar{\Omega}_\rho(x_0)) \subset C^1(\bar{\Omega}_\rho(x_0))$ , 且在  $H^{1,p}(\Omega_\rho(x_0), \lambda)$  中稠密, 引理 4 对  $u \in H^{1,p}(\Omega_\rho, \mathbb{R}^N, \lambda)$  也成立.

由引理 1 得边界上加权 Sobolev-Poincaré 不等式:

$$\left(\int_{\partial_\rho(x_0)} |u|^r \lambda dx\right)^{\frac{1}{r}} \leq C \left(\int_{\Omega_\rho(x_0)} |\nabla u|^2 \lambda dx\right)^{\frac{1}{2}} \quad (7)$$

## 2 $L^p$ 估计

考虑非线性椭圆组:

$$-D_a(A_i^a(x, u, \nabla u)) = B_i(x, u, \nabla u) \quad (8)$$

满足退缩椭圆性条件和增长条件:

$$A_i^a(x, u, \nabla u) D_a u^i \geq \lambda(x) |\nabla u|^2 \quad (9)$$

$$|A_i^a(x, u, \nabla u)| \leq \mu \lambda(x) (|\nabla u| + |u|^{\frac{1}{2-s}} + |f_{i,a}|) \quad (10)$$

$$|B_i(x, u, \nabla u)| \leq \mu \lambda(x) (|\nabla u|^s + |u|^{\frac{s}{2-s}} + |g_i|) \quad (11)$$

这里  $2(1 - \frac{1}{r}) \leq s < 2, f(x) \in L^{p_1}(\Omega, \lambda^{\frac{p_1}{2}}), g(x) \in L^{\frac{p_1}{2-s}}(\Omega, \lambda^{\frac{p_1}{2}}), p_1 > 2$ .

文[3]在  $S = 2 - \epsilon$  条件下,证明(8)弱解梯度  $L^p$  估计时,需弱解有界,本文将去掉这一限制.

**定理 1** 设  $\lambda^2(x) \in A_2$ , 且有界, 条件(9)~(11)成立,  $u$  是方程组(8)的弱解. 则存在  $P \in (2, P_1)$ , 使得  $u \in H_{loc}^{1, \frac{P}{2}}(\Omega, \mathbb{R}^N, \lambda^{\frac{P}{2}})$ . 且当  $R$  充分小时, 有

$$\int_{B_R} (|\nabla u|^2 + |u|^{\frac{S}{2-3}})^{\frac{P}{2}} \lambda^{\frac{P}{2}} dx \leq C \left\{ \int_{B_{2R}} (|\nabla u|^2 + |u|^{\frac{S}{2-3}}) \lambda dx \right\}^{\frac{1}{2}} + \left( \int_{B_{2R}} (|f| + |F|)^p \lambda^{\frac{P}{2}} dx \right)^{\frac{1}{p}}$$

这里常数  $P, C$  仅依赖于  $n, N, P_1$  和  $\mu$ ,

$$F = R \left( \int_{B_{2R}} (|g|^{\frac{r}{r-1}} \lambda dx)^{\frac{1}{2}(1-\frac{2}{r})} |g|^{\frac{r}{2(r-1)}} \right)$$

**证明** 设  $\eta(x)$  是标准截断函数, 取试验函数  $\varphi = (u - u_{2R})\eta^2$ , 利用(9)~(11)得

$$\int_{B_R} |\nabla u|^2 \lambda dx \leq CR^{-2} \int_{B_{2R}} |u - u_{2R}|^2 \lambda dx + C \int_{B_{2R}} |f|^2 \lambda dx + C \int_{B_{2R}} |u|^{\frac{2}{2-3}} \lambda dx + C \int_{B_{2R}} (|\nabla u|^2 + |u|^{\frac{S}{2-3}} + |g|) |u - u_{2R}| \lambda dx \quad (12)$$

由  $\lambda^2(x) \in A_2$ , 知  $\lambda(x) \in A_{\frac{3}{2}}$ , 于是当  $8 \leq 3r$  时, 由(5)

$$R^{-2} \int_{B_{2R}} |u - u_{2R}|^2 \lambda dx \leq C \lambda(B_{2R})^{-\frac{1}{3}} (\lambda^2(B_{2R}))^{\frac{1}{6}} \left( \int_{B_{2R}} |\nabla u|^2 dx \right)^{\frac{5}{6}} \left( \int_{B_{2R}} |\nabla u| \lambda^{\frac{1}{2}} dx \right)^{\frac{1}{3}}$$

由  $\lambda^2(x) \in A_2$ , 知  $\lambda(x) \in A_2$ , 且有

$$(\lambda^\pm(B_R))^2 \leq \omega_n R^n \lambda^2(B_R) \quad (13)$$

$$|B_R|^2 \leq \lambda(B_R) \lambda^{-1}(B_R) \leq L_2 |B_R|^2 \quad (14)$$

利用(13)和(14)得

$$\lambda^2(B_{2R}) \lambda(B_{2R})^{-2} \leq CR^{-3n} \lambda^2(B_{2R}) \lambda^{-2}(B_{2R}) \leq CR^{-n} \quad (15)$$

$$\therefore R^{-2} \int_{B_{2R}} |u - u_{2R}|^2 \lambda dx \leq \frac{1}{4} \int_{B_{2R}} |\nabla u|^2 \lambda dx + C(R^{\frac{n}{2}} \int_{B_{2R}} |\nabla u| \lambda^{\frac{1}{2}} dx)^2 \quad (16)$$

当  $3r \leq 8$  时,  $\lambda(x) \in A_{4/r}$ , 由(5)有

$$R^{-2} \int_{B_{2R}} |u - u_{2R}|^2 \lambda dx \leq \frac{1}{4} \int_{B_{2R}} |\nabla u|^2 \lambda dx + CR^n \left( \int_{B_{2R}} |\nabla u| \lambda^{\frac{1}{2}} \right)^2$$

所以对任一嵌入临界指数  $r$ , (16) 均成立.

$$\int_{B_{2R}} |u|^{\frac{2}{2-3}} \lambda dx \leq C \left( \int_{B_{2R}} |u - u_R|^{\frac{2r}{r-2}, \frac{r-1}{2-3}} \lambda dx \right)^{1-\frac{2}{r}} \int_{B_{2R}} |\nabla u|^2 \lambda dx + C \lambda(B_{2R})^{-\frac{1}{2-3}} \left( \int_{B_{2R}} |u| \lambda dx \right)^{\frac{2}{2-3}}$$

当  $4 > 3s$  时

$$\int_{B_{2R}} |u| \lambda dx \leq \lambda^2(B_{2R})^{\frac{1}{4}} \left( \int_{B_{2R}} (|u|^{\frac{2}{2-3}} \lambda)^{\frac{1}{2}} dx \right)^{\frac{1}{2}} \left( \int_{B_{2R}} |u|^{\frac{2}{2-3}} \lambda dx \right)^{1-\frac{3s}{4}}$$

当  $4 \leq 3s$  时, 有

$$\int_{B_{2R}} |u| \lambda dx \leq \left( \int_{B_{2R}} (|u|^{\frac{2}{2-3}} \lambda)^{\frac{1}{2}} dx \right)^{2-5} R^n \cdot \frac{3S-4}{4} \lambda^2(B_{2R})^{\frac{5}{4}}$$

$$\begin{aligned} \therefore \int_{B_{2R}} |u|^{\frac{2}{2-s}} \lambda dx &\leq C \left( \int_{B_{2R}} |u - u_R|^{\frac{2r}{r-2} \cdot \frac{s-1}{2-s}} \lambda dx \right)^{1-\frac{2}{r}} \int_{B_{2R}} |\nabla u|^2 \lambda dx \\ &+ CR^n \left( \int_{B_{2R}} (|u|^{\frac{2}{2-s}} \lambda)^{\frac{1}{2}} dx \right)^2 \end{aligned} \tag{17}$$

$$\begin{aligned} \int_{B_{2R}} |\nabla u|^s |u - u_{2R}| \lambda dx &\leq \frac{1}{4} \int_{B_{2R}} |\nabla u|^2 \lambda dx \\ &+ \left( \int_{B_{2R}} |u - u_{2R}|^{\frac{2r}{r-2} \cdot \frac{s-1}{2-s}} \lambda dx \right)^{1-\frac{2}{r}} \int_{B_{2R}} |\nabla u|^2 \lambda dx \end{aligned} \tag{18}$$

由 Hölder 不等式及 (5) 有

$$\begin{aligned} \int_{B_{2R}} |u|^{\frac{s}{2-s}} |u - u_{2R}|^2 \lambda dx &\leq \int_{B_{2R}} |u|^{\frac{2}{2-s}} \lambda dx \\ &+ C \left( \int_{B_{2R}} |u - u_{2R}|^{\frac{2r}{r-2} \cdot \frac{s-1}{2-s}} \lambda dx \right)^{1-\frac{2}{r}} \int_{B_{2R}} |\nabla u|^2 \lambda dx \\ \int_{B_{2R}} |g| |u - u_{2R}| \lambda dx &\leq \frac{1}{4} \int_{B_{2R}} |\nabla u|^2 \lambda dx \\ &+ CR^{2n(1-\frac{1}{r})} \left( \int_{B_{2R}} |g|^{\frac{r}{r-1}} \lambda dx \right)^{2(1-\frac{1}{r})} \end{aligned} \tag{19}$$

把(16) ~ (19) 代入(12), 然后两边加(17), 当  $R$  适当小时, 有

$$\begin{aligned} \int_{B_R} (|\nabla u|^s + |u|^{\frac{2}{2-s}} \lambda dx) &\leq C \left( \int_{B_{2R}} |\nabla u| + |u|^{\frac{1}{2-s}} \lambda^{\frac{1}{2}} dx \right)^2 \\ &+ \left( \int_{B_{2R}} |f|^2 \lambda dx + R^{n(1-\frac{2}{r})} \left( \int_{B_{2R}} |g|^{\frac{r}{r-1}} \lambda dx \right)^{2(1-\frac{2}{r})} + \theta \int_{B_{2R}} (|\nabla u|^2 + |u|^{\frac{2}{2-s}} \lambda dx) \right) \end{aligned}$$

这里  $0 < \theta < 1$ , 根据[5] 第五章命题 1.1, 定理得证.

### 3 内部正则性

考虑拟线性椭圆组

$$-D_\alpha (A_{ij}^{\alpha\beta}(x, u) D_\beta u^j + a_i^\alpha(x, u)) = B_i(x, u, \nabla u) \tag{20}$$

满足退缩椭圆条件:

$$\begin{aligned} A_{ij}^{\alpha\beta}(x, u(x)) &= a^{\alpha\beta}(x, u(x)) \delta_{ij} + B_{ij}^{\alpha\beta}(x, u(x)) \\ \lambda(x) |\xi|^2 &\leq a^{\alpha\beta}(x, u(x)) \xi_\alpha \xi_\beta \leq \Lambda \lambda(x) |\xi|^2 \quad \forall \xi \in \mathbb{R}^N \end{aligned} \tag{21}$$

$$- \tau \lambda(x) |\xi|^2 \leq B_{ij}^{\alpha\beta}(x, u(x)) \xi_\alpha \xi_\beta \leq \tau \lambda(x) |\xi|^2 \quad \forall \xi \in \mathbb{R}^{nN} \tag{22}$$

这里  $\tau$  适当小. 且  $B_i(x, u, \nabla u)$  满足条件 (11) 及

$$|a_i^\alpha(x, u(x))| \leq \mu \lambda(x) (|u|^{\frac{1}{2-s}} + |f_{i,\alpha}(x)|), \quad 2(1 - \frac{1}{r}) \leq S < 2 \tag{23}$$

$f \in L^q(\Omega, \lambda), g \in L^{\frac{q}{2}}(\Omega, \lambda), q > 2n$

可说这是一类主对角线占优的椭圆组. 值得指出的是对角形退缩椭圆组是它的一个特例.

**定理 2** 设  $\lambda^2(x) \in A_2$ , 且有界, 条件(21) ~ (23) 和(11) 成立,  $u \in H^1 \cap L^{2n \cdot \frac{s-1}{2-s}}(\Omega, \mathbb{R}^N, \lambda)$  是椭圆组(20) 的弱解. 则  $u \in C_{loc}^{0,\alpha}(\Omega, \mathbb{R}^N)$ .

**证明** 令  $a^{\alpha\beta}(x) = a^{\alpha\beta}(x, u(x))$ , 设  $v(x)$  是问题:

$$\begin{cases} -D_a(a^{\alpha\beta}(x)D_\beta u^i) = 0 \\ u^i - u^i \in H_0^1(B_R, \lambda) \end{cases} \quad i = 1, 2, \dots, N \quad (24)$$

的弱解. 根据[4]知, 存在  $\delta \in (0, 1)$ , 使对  $\forall \rho \in (0, R/32)$

$$\int_{B_\rho} |\nabla u|^2 \lambda dx \leq C \frac{\lambda(B_{2\rho})}{\lambda(B_R)} \int_{B_R} |\nabla u|^2 \lambda dx \cdot \left(\frac{\rho}{R}\right)^{2\delta-2}$$

利用 (15) 有

$$\int_{B_\rho} |\nabla V|^2 \lambda dx \leq C \left(\frac{\rho}{R}\right)^{2n-2+2\delta} \int_{B_R} |\nabla u|^2 \lambda dx \quad (25)$$

如果  $R/32 \leq \rho < R$ , (25) 是平凡的. 分解  $u = w + v$ , 显然  $W^i$  满足方程

$$-D_a(a^{\alpha\beta} D_\beta w^i) = D_a(B_{ij}^{\alpha\beta} D_\beta u^j + a_i^?) + B_i(x, u, \nabla u)$$

取  $\psi_i = w^i$ , 利用 (21) - (23) 和 (11) 有

$$\begin{aligned} \int_{B_R} |\nabla w|^2 \lambda dx &\leq C \int_{B_R} (|u|^{\frac{2}{2-i}} + |f|^2) \lambda dx + C\tau^2 \int_{B_R} |\nabla u|^2 \lambda dx \\ &\quad + C \int_{B_R} (|\nabla u|^i + |u|^{\frac{2}{2-i}} + |g|) |w| \lambda dx \end{aligned} \quad (26)$$

利用 (4), Hölder 不等式, 并注意到  $w \in H^1 \cap L^{2n, \frac{2}{2-i}}(\Omega, \mathbb{R}^N, \lambda)$ , 由 Lebesgue 积分的绝对连续性, 知任给  $\epsilon > 0$  存在  $R_1$ , 使当  $R < R_1$  时, 有

$$\int_{B_R} |\nabla u|^i |w| \lambda dx \leq \epsilon \int_{B_R} |\nabla u|^2 \lambda dx + \frac{1}{3} \int_{B_R} |\nabla w|^2 \lambda dx \quad (27)$$

$$\int_{B_R} |u|^{\frac{2}{2-i}} |w| \lambda dx \leq C \int_{B_R} |u|^{\frac{2}{2-i}} \lambda dx + \epsilon \int_{B_R} |\nabla w|^2 \lambda dx \quad (28)$$

$$\int_{B_R} |g| |w| \lambda dx \leq \frac{1}{3} \int_{B_R} |\nabla w|^2 \lambda dx + C\lambda^{-1}(B_R) \left( \int_{B_R} |g|^{\frac{r}{r-1}} \lambda dx \right)^{2(1-\frac{1}{r})} \quad (29)$$

利用 (15), 并注意到  $1 - \frac{2}{r} - \frac{2}{q} > 0$ , 记  $a = 1 - \frac{2n}{q}$ , 得

$$\int_{B_R} |f|^2 \lambda dx \leq \left( \int_{B_R} |f|^q \lambda dx \right)^{\frac{1}{q}} \lambda^{-1}(B_R) \lambda(B_R)^{1-\frac{2}{q}} \leq CR^{2n-2+2a} \quad (30)$$

$$\lambda^{-1}(B_R) \left( \int_{B_R} |g|^{\frac{r}{r-1}} \lambda dx \right)^{2(1-\frac{1}{r})} \leq C\lambda^{-1}(B_R) \lambda(B_R)^{1-\frac{2}{q}} \leq CR^{2n-2+2a} \quad (31)$$

当  $R < R_1$  时, 由 (25) - (31) 得

$$\phi(\rho) \leq C \left[ \left(\frac{\rho}{R}\right)^{2n-2+2\delta} + \epsilon + \tau^2 \right] \phi(R) + C \int_{B_R} |u|^{\frac{2}{2-i}} \lambda dx + CR^{2n-2+2a} \quad (32)$$

$$\begin{aligned} \int_{B_\rho} |u|^{\frac{2}{2-i}} \lambda dx &\leq C \int_{B_R} |u - u_R|^{\frac{2}{2-i}} \lambda dx + C |u_\rho|^{\frac{2}{2-i}} \lambda(B_\rho) \lambda^{-1}(B_\rho) \\ &\leq CR^\beta \int_{B_R} |\nabla u|^2 \lambda dx + C \left(\frac{\rho}{R}\right)^{2n} \int_{B_R} |u|^{\frac{2}{2-i}} \lambda dx \end{aligned} \quad (33)$$

这里  $\beta = n(1 - \frac{2}{r} - \frac{1}{n})(\frac{2}{s} - 1)$ . 记  $\theta = (1 - \frac{2}{p})n$ , 由定理 1 知, 存在  $P > 2$ , 对  $2\rho < R_0 = \text{dist}(x_0, \partial\Omega)$ , 有

$$\int_{B_\rho} (|\nabla u| + |u|^{\frac{2}{2-\tau}}) \lambda dx \leq \left( \int_{B_{R_0}} (|\nabla u| + |u|^{\frac{2}{2-\tau}}) \lambda^{\frac{1}{2}} dx \right)^{\frac{2}{2-\tau}} \rho^{(1-\frac{2}{2-\tau})n} \leq C\rho^\theta \quad (34)$$

把(34)代入(33),注意到 $\lambda^{-1}(B_R)$ 有界,若 $\beta + \theta \leq 2n - 2 + 2a$ 时,利用[5]第 III 章引理 2.1,得

$$\int_{B_\rho} |u|^{\frac{2}{2-\tau}} \lambda dx \leq C\rho^{\beta+\theta} \quad (35)$$

不妨设 $\alpha < \delta$ ,结合(32)和(34),当 $\tau$ 适当小时,利用同样引理知,存在 $R_2$ 使当 $R < R_2$ 时,有

$$\phi(\rho) \leq C\rho^{\beta+\theta} \quad (36)$$

与(35),(36)推导一样,得

$$\int_{B_\rho} |u|^{\frac{2}{2-\tau}} \lambda dx \leq C\rho^{2\beta+\theta}$$

$$\phi(\rho) \leq C\rho^{2\beta+\theta}$$

经过 $m$ 步迭代后,当 $m$ 满足

$$(m-1)\beta + \theta < 2n - 2 + 2a \leq m\beta + \theta$$

由于 $\tau$ 充分小,所以当 $R < R_2$ 时,有

$$\phi(\rho) \leq C \left[ \left( \frac{\rho}{R} \right)^{2n-2+2a} \phi(R) + \rho^{2n-2+2a} \right] \leq C\rho^{2n-2+2a} \quad (37)$$

$$\int_{B_\rho} |\nabla u| dx \leq (\phi(\rho))^{\frac{1}{2}} \leq C\rho^{n-1+a}$$

根据 Morrey 定理 [5], 定理得证.

## 4 边界正则性

考虑(20)的 Dirichlet 问题, 边值函数 $h(x) \in H^{1,p}(\Omega, \mathbb{R}^N, \lambda)$ , 令 $U = u - h$ , 则 $U$ 的边值为零, 故不妨设, 在 $\partial\Omega$ 上, $h(x) = 0$ . 设 $\partial\Omega \in C^1$ , 则存在微分同胚

$$G: \Omega_R(x_0) \rightarrow B_2^+(0) \quad \text{且} \quad \partial\Omega \cap B_R \rightarrow \Gamma_2(0)$$

由覆盖定理知, 只需考虑满足

$$\begin{cases} -D_\alpha(A_{ij}^{\alpha\beta}(x,u)D_\beta u^j + a_i^\alpha(x,u)) = B_i(x,u,\nabla u) & x \in B_2^+(0) \\ u = 0 & x \in \Gamma_2(0) \end{cases} \quad (38)$$

的 $u$ 在 $B_1^+ \cup \Gamma_1$ 上的正则性.

**引理 5** 设 $\lambda(x) \in A_2, \nu \in H^1(B_R^+, \mathbb{R}^N, \lambda)$  满足

$$\begin{cases} \int_{B_R^+} a^{\alpha\beta}(x) D_\beta \nu^j D_\alpha \phi dx = 0, \quad \forall \phi \in H_0^1(B_R^+, \mathbb{R}^N, \lambda) \\ \nu|_{\Gamma_R(x_0)} = 0 & i = 1, 2, \dots, N \end{cases} \quad (39)$$

这里 $x_0 \in \Gamma_1, a^{\alpha\beta}(x)$ 满足(21), 则存在 $C = C(\wedge, n, N)$ 和 $\delta \in (0, 1)$ , 使对 $\forall \rho < R$ 有

$$\int_{B_\rho^+} |\nabla \nu|^2 \lambda dx \leq c \left( \frac{\rho}{R} \right)^{2n-2+2\delta} \int_{B_\rho^+} |\nabla \nu|^2 \lambda dx \quad (40)$$

**证明** 对  $\forall 0 < \rho < R/32$ , 取  $\varphi = (v - v_p^+) \eta^2$ , 由(39)可得边界上加权 Caccioppoli 不等式, 再利用 [1] 定理 2、4、6 和 (7), 得

$$\int_{B_\rho^+} |\nabla v|^2 \lambda dx \leq C \rho^{-2} \frac{\lambda(B_{2\rho})}{\lambda(B_R)} \int_{B_R} v^2 \lambda dx \cdot \left(\frac{2\rho}{R}\right)^{2\theta}$$

由(15)得(40). 如果  $R/32 \leq \rho < R$ , (40) 是平凡的.

**定理 4** 设  $\lambda^2(x) \in A_2$ , 且有界, 条件(21)–(23)和(11)成立,  $u \in H^1 \cap L^{2n \frac{S-1}{S-3}}(B_2^+, \mathbb{R}^N, \lambda)$  满足(38), 且  $u|_{\Gamma_2(0)} = 0$ . 则  $u \in C^{0,\alpha}(B_1^+ \cup \Gamma_1, \mathbb{R}^N)$ .

**证明** 设  $x_0 \in \Gamma_1$  和  $R < 1$ ,  $v(x)$  是 Dirichlet 问题

$$\begin{cases} \int_{B_R^+} a^{i\rho}(x) D_\alpha v^i D_\beta \phi dx = 0, & \forall \phi \in H_0^1(B_R^+, \mathbb{R}^N, \lambda) \\ v^i - u^i \in H_0^1(B_R, \lambda) & i = 1, 2, \dots, N \end{cases} \quad (41)$$

的解. 由(38)和(41), 对  $\forall \rho \in (0, R)$ , 有

$$\int_{B_\rho^+} |\nabla v|^2 \lambda dx \leq C \left(\frac{\rho}{R}\right)^{2n-2+2\theta} \int_{B_\rho^+} |\nabla u|^2 \lambda dx \quad (42)$$

类似(32)推导, 存在  $R_2$ , 当  $R < R_2$  时, 有

$$\begin{aligned} \int_{B_\rho^+} |\nabla v|^2 \lambda dx &\leq C \left[ \left(\frac{\rho}{R}\right)^{2n-2+2a} + \tau^2 + \varepsilon \right] \int_{B_\rho^+} |\nabla u|^2 \lambda dx \\ &+ C \int_{B_\rho^+} |u|^{\frac{2}{2-\beta}} \lambda dx + CR^{2n-2+2a} \end{aligned} \quad (43)$$

由(7)得

$$\int_{B_\rho^+} |u|^{\frac{2}{2-\beta}} \lambda dx + \leq C \left( \int_{B_R^+} |u|^{\frac{2r}{2-\beta} \frac{S-1}{S-3}} \lambda dx \right)^{1-\frac{2}{r}} \int_{B_\rho^+} |\nabla u|^2 \lambda dx \quad (44)$$

把(44)代入(43), 由 Lebesgue 积分的绝对连续性, 当  $\tau$  足够小时, 利用 [5] 第 III 章引理 2.1 知, 对  $\forall x_0 \in \Gamma_1$ , 存在  $R_3$ , 使得对  $0 < \rho < R < R_3$ , 有

$$\phi(x_0, \rho) \leq C \left[ \left(\frac{\rho}{R}\right)^{2n-2+2a} \phi(x_0, R) + \rho^{2n-2+2a} \right] \quad (45)$$

我们断言: 对  $\forall x_1 \in B_1^+ \cup \Gamma_1$ , 存在  $R$ , 使得对所有  $\rho < R$ , 有

$$\phi(x_1, \rho) \leq C \left[ \left(\frac{\rho}{R}\right)^{2n-2+2a} \phi(x_1, R) + \rho^{2n-2+2a} \right] \quad (46)$$

事实上, 设  $x_1 \in B_1^+$ , 选  $x_0 \in \Gamma_1$ , 使  $d = \text{dist}(x_1, \Gamma_1) = |x_1 - x_0|$ , 取  $R \leq \min(R_1, R_3, 1)$ .

对  $\rho \geq R/8$ , (46) 是平凡的. 对  $\rho < R/8$ , 分两种情形讨论

(i) 若  $R/8 \leq d$ , 则  $B_{R/8} \subset B_1^+$ , 利用(37)有

$$\phi(x_1, \rho) \leq C \left[ \left(\frac{\rho}{R}\right)^{2n-2+2a} \phi(x_1, R) + \rho^{2n-2+2a} \right]$$

(ii) 若  $R/8 \geq d$ , 又分两种情形:

a) 当  $\rho \geq d/2$  时

$$B_\rho^+(x_1) \subset B_{4\rho}^+(x_0) \subset B_{R/2}^+(x_0) \subset B_\rho^+(x_1)$$

由(45)得  $\phi(x_1, \rho) \leq C \left[ \left(\frac{\rho}{R}\right)^{2n-2+2a} \phi(X_1, \rho) + \rho^{2n-2+2a} \right]$

b) 当  $\rho < d/2$  时

$$B_\rho(x_1) \subset B_{d/2}(x_1) \subset B_{2d}^+(x_0) \subset B_{R/2}^+(x_0) \subset B_\rho^1(x_1)$$

利用 (37) 和 (45) 有

$$\phi(x_1, \rho) \leq C \left[ \left( \frac{\rho}{R} \right)^{2n-2+2a} \phi(x_1, R) + \rho^{2n-2+2a} \right]$$

由零延拓和 Morrey 定理, 定理得证.

附注: (1) 如果增长条件改为:

$$|a_i^a(x, u)| \leq \mu\lambda(x) (|u|^{\frac{r}{2}} + |f_{i,a}|)$$

$$|B_i(x, u, p)| \leq \mu\lambda(x) (|p|^r + |u|^{r-1} |g_i|)$$

则可证明  $u \in H^1(\Omega, \mathbb{R}^N, \lambda)$  定理结论成立.

(2) 如果  $u$  有界, 则条件  $\lambda^2(x) \in A_2$  可改善为  $\lambda(x) \in A_2$ .

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## Global Regularity of Weak Solutions to a Class of Degenerate Quasilinear Elliptic Systems

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**Abstract** Under growth condition, we prove the  $L^p$ -estimate for the gradient of weak solutions to a class of degenerate nonlinear elliptic systems and the global regularity of weak solutions to a class of degenerate quasilinear elliptic systems.

**Keywords** Degenerate elliptic systems,  $L^p$ -estimate, Global regularity

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