第36卷 第6期 中山大学学报 (自然科学版) Vol. 36 No. 6
1997年 11月 ACTA SCIENTIARUM NATURALIUM UNIVERSITATIS SUNYATSENI Nov. 1997

On the Generalized Kloosterman Sums*

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Keywords Kloosterman sums, dirichlet character, primitive character
Classificational Number O 156.4

Let q be a positive integer, $\chi \bmod q$ be the Dirichlet character as usual. If a and b are the integers, the Kloosterman sums with character $\chi \bmod q$ is defined by

$$S_\chi(a,b,q)=\sum_{x \bmod q} \chi(x)\, e\left(\frac{ax+b\bar{x}}{q}\right) \tag{1}$$

where $x\bar{x}\equiv 1 \pmod q$ and $e(\theta)=e^{2\pi i\theta}$ for real θ. If $\chi=\chi_0$ is the trivial character, then $S_{\chi_0}(a, b, q)$ is known as the classical Kloosterman sums and we have Weil-Esterman type bound as follows

$$|S_{\chi_0}(a, b, q)| \leqslant d(q)\, q^{\frac{1}{2}}\, (a, b, q)^{\frac{1}{2}} \tag{2}$$

where $d(q)$ is the number of positive divisors of q. For the generalized Kloosterman sums, some scholars such as. Selberg[1], Iwaniec[2] and Duke, Friedlander and Iwaniec[3] used the same estimate for $S_\chi(a, b, q)$.

This paper proves the Weil-Esterman type bound is not correct for the generalized Kloosterman sums, and gets the condition for the estimate.

To state the main result, we need a notation of local order for the character $\chi \bmod q$. For any character χ, there exists a smallest positive integer n such that $\chi^n=\chi_0$, we call n is the order of χ. Let $q=p_1^{\alpha_1}\cdots p_s^{\alpha_s}$, then $\chi=\chi_1\chi_2\cdots\chi_s$, where $\chi_j \bmod q_j^{\alpha_j}$ is the character, it is easy to see that ord $\chi_j \mid \varphi(p_j^{\alpha_j})=p_j^{\alpha_j-1}(p_j-1)$, so we define the local order of χ on $p_j^{\alpha_j}$ by the following formula

$$N(\chi, p_j^{\alpha_j}) = (\text{ord}\, \chi_j,\ p_j^{\alpha_j}) \tag{3}$$

The main results of this paper is the following theorem.

Theorem Let q be a positive integer, $\chi \bmod q$ be the Dirichlet character, then we have

(i) If $q=p_1p_2\cdots p_s$ is a square-free number, then for all $S_\chi(a, b, q)$ we have uniformly

$$|S_\chi(a, b, q)| \leqslant d(q)\, q^{\frac{1}{2}}\, (a, b, q)^{\frac{1}{2}} \tag{4}$$

* 国家杰出青年基金 (19625102) 资助项目
收稿日期: 1997-10-06 郑志勇，男，34岁，教授

(ii) For all $p_j^{T_j}\|q$ with $T_j > 1$, if $N(i, p_j^{T_j})(a,b,p_j^{T_j}) < p_j^{T_j-1}$, then we have

$$|S_i(a,b,q)| \leqslant d(q)\ q^{\frac{1}{2}}(a,b,q)^{\frac{1}{2}} \quad (5)$$

(iii) If $i \neq i_0$ and there exists $p_j^{T_j}\|q$ such that $N(i, p_j^{T_j})(a,b,p_j^{T_j}) \geqslant p_j^{T_j}$, then

$$S_i(a,b,q) = 0 \quad (6)$$

(iv) If $(a,b,q) > 1$, then for all i mod q primitive we have

$$S_i(a,b,q) = 0 \quad (7)$$

(v) For any integers a and b, there exist many infinite q such that $(a,b,q) = 1$ and exist at least a i mod q primitive such that

$$|S_i(a,b,q)| \geqslant (1 - 1/\sqrt{2})q^{\frac{2}{3}} \quad (8)$$

The conclusion (i) of theorem is due to Chowla[4], in which he proved that Weil bound is true for all $S_i(a,b,q)$ when q is prime number or this sums is defined over any finite fields.

Acknowledgment The author is very grateful to Professor Deng Donggao, for his kind help and encouragement.

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关于一般的 Kloosterman和

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摘 要 在整数环上研究一般的 Kloosterman和,给出其下界估计,否定了 Iwaniec等人的上界结果,同时在一定条件下证明了 Weil-Esterman上界的存在时,将 Kloosterman和与 Salié 和的经典结果进一步予以扩张.

关键词 Kloosterman和, Dirichlet特征,本原特征

分类号 O 156.4

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