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一类孤子族的 Lax表示*

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摘 要 对一类特征值问题定义映射 e给出一个 Lax对 U, V_m,其相应的 Lenard序列 $K a_{j-1} = J a_j$ 不含逆微分算子 ∂^{-1},可显式求出线性表示的通解 $a_m = E c_s g_{m-s}$,从而得到了一类显式表示的孤子族 $T_{t_m} = X_m$.

关键词 孤子方程, Lax表示 ,符号计算

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众所周知,利用 Lax对可将特征值问题和孤子方程联系起来 . 通过 Lenard递推序列更可得到一组孤子方程族 . 利用这方法已有一系列文章得到了丰富的结果 [1-3]. 可惜的是解 Lenard方程时往往含有逆微分算子 ∂^{-1},不易确定是否可得显式解,是否有显式表示的孤子方程族 .

本文讨论一类特征值问题 [4]

$$H_x = UH,\quad U = \lambda\begin{pmatrix} u & v+\lambda \\ v-\lambda & -u \end{pmatrix} \tag{1}$$

通过定义映射 e给出一个 Lax对 U, V_m. 其对应的 Lenard递推序列方程不含逆微分算子 ∂^{-1},可具体显式求出其通解 . 同时证明了其通解可通过某特解线性递推表示,从而得到了显式表示的一类孤子方程族 . 文中应用吴文俊数学机械化理论中的消元方法,并应用了 Mathematica™数学软件编制的符号计算程序自动递推孤子方程族 .

考虑特征方程 (1),令 $U = \lambda e(T)$,其中

$$e(T) = \begin{pmatrix} T^1 & T^2 - T^3 \\ T^2 + T^3 & -T^1 \end{pmatrix},\qquad T = (T^1, T^2, T^3)^{\mathrm{T}}$$

则对 (1)式有 $U = \lambda e(T)$, $T = (u, v, -\lambda)^{\mathrm{T}}$. 取 $V = \lambda e(U)$, $U = (U^1, U^2, U^3)^{\mathrm{T}}$, 于是

$$[U, V] = \lambda^2 [e(T), e(U)] = \lambda^2 e(2T T \times U) \equiv \lambda^2 e(AU) \tag{2}$$

其中, $T = \mathrm{diag}(1, 1, -1)$, $A = \begin{pmatrix} 0 & 2\lambda & 2v \\ -2\lambda & 0 & -2u \\ 2v & -2u & 0 \end{pmatrix}$

记 $U = SG$, $S = \mathrm{diag}(1, 1, \lambda)$, $G = (G^1, G^2, G^3)^{\mathrm{T}}$, $\partial = \dfrac{\partial}{\partial x}$, 则

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$$V_x - [U,V] = \lambda^{e}((\partial - \lambda A)U) = \lambda^{e}(S(K-\lambda^2 J)G) \tag{3}$$

这里 $K=\begin{pmatrix}\partial & 0 & 0\\ 0 & \partial & 0\\ -2v & 2u & \partial\end{pmatrix}$, $J=\begin{pmatrix}0 & 2 & 2v\\ -2 & 0 & -2u\\ 0 & 0 & 0\end{pmatrix}$

定义 Lenard序列 $\{a_j\}$:

$$J a_0 = 0,\quad K a_{j-1} = J a_j \qquad (j=1,2,\cdots) \tag{4}$$

定理 1 方程(4)有通解: $a_0 = e_0(u,v,-1)^{T}$

$$a_1 = \left(\frac{1}{2}e_0 v_x + \frac{1}{2}e_0 u(u^2+v^2) + e_1 u,\ \frac{1}{2}e_0 v(u^2+v^2) + e_1 v,\ -\frac{1}{2}e_0(u^2+v^2) - e_1\right)^{T}$$

$$\begin{cases} a^1_{m+1} = -\frac{1}{2}\partial a^2_m - u a^3_{m+1} \\ a^2_{m+1} = \frac{1}{2}\partial a^1_m - v a^3_{m+1} \qquad (m=1,2,\cdots) \\ a^3_{m+1} = -\frac{1}{2}u a^1_m - \frac{1}{2}v a^2_m - \frac{1}{2}\sum_{j=1}^{m}\bar{e}(a^1_j a^1_{m-j} + a^2_j a^2_{m-j} - a^3_j a^3_{m+1-j}) - e_{m+1} \end{cases} \tag{5}$$

其中, $e_0, e_1, \cdots, e_{m+1}, \cdots$为任意常数, $e_0 \neq 0$, $\bar{e} = e_0^{-1}$.

证明 a_0可直接验证. 将(4)改写为

$$\begin{cases} a^1_{m+1} = -\frac{1}{2}\partial a^2_m - u a^3_{m+1} \\ a^2_{m+1} = \frac{1}{2}\partial a^1_m - v a^3_{m+1} \\ \partial a^3_m = 2v a^1_m - 2u a^2_m \end{cases} \tag{6}$$

(5)式的前两分式与(6)式相同. 现证 a_1和(5)的第三分式成立.

记 $f_m = (a^1_m, a^2_m)^{T}$, $\bar{J} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$, $E = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $a * b = a^1 b^1 + a^2 b^2$.

于是, $f_0 = e_0(u,v)^{T}$, $a * b = b * a$, $\bar{J}a * b = -a * \bar{J}b$, $\bar{J}a * a = 0$, $\bar{J}a * \bar{J}b = a * b$, $\bar{J}\bar{J} = E$.

(6)式可化为

$$\begin{cases} f_{m+1} = \frac{1}{2}\bar{J}\partial f_m - \bar{e}a^3_{m+1} f_0 \\ \partial f_m = -2\bar{J}f_{m+1} - 2\bar{e}a^3_{m+1}\bar{J}f_0 \\ \partial a^3_m = -2\bar{e}\bar{J}f_0 * f_m = 2\bar{e}f_0 * \bar{J}f_m \end{cases} \tag{7}$$

反复应用上述关系式,得

$$\bar{J}f_j * f_{m+1-j} + \bar{e}a^3_j \bar{J}f_0 * f_{m+1-j} = \bar{J}f_j * \left(\frac{1}{2}\bar{J}\partial f_{m-j} - \bar{e}a^3_{m+1-j}f_0\right) - \frac{1}{2}a^3_j \partial a^3_{m+1-j} =$$

$$\frac{1}{2}f_j * \partial f_{m-j} - a^3_{m+1-j}\bar{J}f_j * f_0 - \frac{1}{2}a^3_j \partial a^3_{m+1-j} =$$

$$\frac{1}{2}f_j * \partial f_{m-j} - \frac{1}{2}\partial a^3_j a^3_{m-j} - \frac{1}{2}a^3_j \partial a^3_{m+1-j} =$$

$$\frac{1}{2}\partial(f_j * f_{m-j} - a^3_j a^3_{m-j}) - \frac{1}{2}\partial f_j * f_{m-j} =$$

$$\frac{1}{2}\partial(f_j * f_{m-j} - a^3_j a^3_{m-j}) + \bar{J}f_{j+1} * f_{m-j} + \bar{e}a^3_{j+1}\bar{J}f_0 * f_{m-j}$$

于是

$$Jf_0 * f_{m+1} + e f_0^3 J f_0 * f_{m+1} = \frac{1}{2}\sum_{j=0}^{m} \partial(f_j * f_{m-j} - f_j^3 f_{m+1-j}^3) + J f_{m+1} * f_0 + e f_{m+1}^3 J f_0 * f_0$$

利用 $f_0^3 = -e_0$, $Jf_0 * f_0 = 0$ 及(7)式,有 $\partial f_{m+1}^3 = -\sum_{j=0}^{m} \bar{e}(f_j * f_{m-j} - f_j^3 f_{m+1-j}^3)$,
即得通解

$$f_{m+1}^3 = -\sum_{j=0}^{m} \bar{e}(f_j * f_{m-j} - f_j^3 f_{m+1-j}^3) - 2e_{m+1} \tag{8}$$

其中, e_{m+1}为任意常数. 整理之,有

$$f_1^3 = -\frac{1}{2}\bar{e} f_0 * f_0 - e_1 = -\frac{1}{2} e_0 (u^2 + v^2) - e_1$$

$$f_{m+1}^3 = -\frac{1}{2}\bar{e} f_0 * f_m - \frac{1}{2}\sum_{j=1}^{m} \bar{e}(f_j * f_{m-j} - f_j^3 f_{m+1-j}^3) - e_{m+1} =$$

$$-\frac{1}{2} u f_m^1 - \frac{1}{2} v f_m^2 - \frac{1}{2}\sum_{j=1}^{m} e(f_j^1 f_{m-j}^1 + f_j^2 f_{m-j}^2 - f_j^3 f_{m+1-j}^3) - e_{m+1}$$

证得 f_1^3 及(5)式的第三分式成立. 定理证毕.

方程(4)的通解(5)中取 $e_0 = \bar{e} = 1$, $e_j = 0 (j > 0)$,则得

推论 方程(4)有特解 $g_0 = (u, v, -1)^T$

$$g_1 = \left(-\frac{1}{2} v_x + \frac{1}{2} u (u^2 + v^2), \frac{1}{2} u_x + \frac{1}{2} v (u^2 + v^2), -\frac{1}{2}(u^2 + v^2)\right)^T$$

$$\begin{cases} g_{m+1}^1 = -\frac{1}{2}\partial g_m^2 - u g_{m+1}^3 \\ g_{m+1}^2 = \frac{1}{2}\partial g_m^1 - v g_{m+1}^3 \quad (m = 1, 2, \cdots) \\ g_{m+1}^3 = \frac{1}{2} u g_m^1 - \frac{1}{2} v g_m^2 - \frac{1}{2}\sum_{j=1}^{m} (g_j^1 g_{m-j}^1 + g_j^2 g_{m-j}^2 - g_j^3 g_{m+1-j}^3) \end{cases} \tag{9}$$

定理 2 方程(4)的通解(5)可由特解(9)表示为

$$f_m = c_0 g_m + c_1 g_{m-1} + \cdots + c_m g_0 \quad (m = 0, 1, \cdots) \tag{10}$$

其中, $c_0, c_1, \cdots, c_m, \cdots$为任意(积分)常数, $c_0 \neq 0$. c_i 与 e_i 有关系式

$$c_0 = e_0, \; c_1 = e_1, \; c_m = e_m - \frac{1}{2} c_0^{-1} \sum_{s=1}^{m-1} c_s c_{m-s} \quad (m = 2, 3, \cdots) \tag{11}$$

证明 记 $\bar{g}_m = (g_m^1, g_m^2)^T$,其中 g_m 为特解(9). 应用定理 1中证明的记号,式(8)变为

$$g_{m+1}^3 = -\sum_{j=0}^{m} (\bar{g}_j * \bar{g}_{m-j} - g_j^3 g_{m+1-j}^3) \quad (m = 0, 1, \cdots) \tag{12}$$

显然, $m = 0, 1$时(10)式成立, $c_0 = e_0$, $c_1 = e_1$. 假设(10), (11)式当 $m = 0, 1, \cdots, n$时成立. 则对任意常数 c_{n+1}. 定义

$$f_{n+1} = c_0 g_{n+1} + c_1 g_n + \cdots + c_{n+1} g_0 \tag{13}$$

利用 $Jg_0 = 0$及 g_j 满足的关系式(4)有

$$K f_n = c_0 K g_n + c_1 K g_{n-1} + \cdots + c_n K g_0 = c_0 J g_{n+1} + c_1 J g_n + \cdots + c_n J g_1 + c_{n+1} J g_0 = J f_{n+1}$$

即 f_{n+1}满足 Lenard方程(4),应用定理 1的通解式(8),将 $m = 0, 1, \cdots, n$时的(10)式代入,重新整理,同时利用关系式(12),可得

$$f_{n+1}^3 = -\frac{1}{2}\bar{e} f_0 * f_n - \frac{1}{2}\sum_{j=1}^{n} \bar{e}(f_j * f_{n-j} - f_j^3 f_{n+1-j}^3) - e_{n+1} =$$

$$-\frac{1}{2}\bar{g}_0 * \Big(\sum_{s=0}^{n} c_s\bar{g}_{n-s}\Big) - \frac{1}{2}\sum_{j=1}^{n}\Big(\Big(\sum_{e=0}^{j} c_s\bar{g}_{j-s}\Big) * \Big(\sum_{e=0}^{n-j} c_e\bar{g}_{n-j-e}\Big) - \Big(\sum_{s=0}^{j} c_s g^{3}_{j-s}\Big)\circ$$

$$\Big(\sum_{e=0}^{n+1-j} c_e g^{3}_{n+1-j-e}\Big)\Big) - e_{n+1} =$$

$$-\frac{1}{2}\sum_{s=0}^{n} c_s\bar{g}_0 * \bar{g}_{n-s} - \frac{1}{2}\sum_{j=1}^{n}\sum_{s=0}^{j}\sum_{e=0}^{n-j} c_s c_e(\bar{g}_{j-s} * \bar{g}_{n-j-e} - g^{3}_{j-s}g^{3}_{n+1-j-e}) +$$

$$\frac{1}{2}\sum_{j=1}^{n}\sum_{e=0}^{j} c_s c_{n+1-j} g^{3}_{j-s} g^{3}_0 - e_{n+1} =$$

$$-\sum_{s=0}^{n}\frac{1}{2}c_s\bar{g}_0 * \bar{g}_{n-s} - \frac{1}{2}\sum_{e=0}^{n-1} c_e c\sum_{j=0}^{n-e}(\bar{g}_j * \bar{g}_{n-e-j} - g^{3}_j g^{3}_{n+1-e-j}) -$$

$$\frac{1}{2}\sum_{s=1}^{n}\sum_{e=0}^{n-s} c_s c_e\sum_{j=0}^{n-e-s}(\bar{g}_j * \bar{g}_{n-e-j} - g^{3}_j g^{3}_{n+1-s-e-j}) + \frac{1}{2}\sum_{s=1}^{n}\sum_{e=0}^{n+1-s} c_s c_e g^{3}_0 g^{3}_{n+1-s-e} - e_{n+1} =$$

$$\sum_{e=0}^{n-1} c_e\Big(-\frac{1}{2}\bar{g}_0 * \bar{g}_{n-e} - \frac{1}{2}\sum_{j=0}^{n-e}(\bar{g}_j * \bar{g}_{n-s-e-j} - g^{3}_j g^{3}_{n+1-e-j})\Big) - \frac{1}{2}c_n\bar{g}_0 * \bar{g}_0 +$$

$$\frac{1}{2}\sum_{s=1}^{n}\sum_{e=0}^{n-s} c_s c_e g^{3}_{n+1-e-s} - \frac{1}{2}\sum_{s=1}^{n}\sum_{e=0}^{n+1-s} c_s c_e g^{3}_{n+1-s-e} - e_{n+1} =$$

$$\sum_{e=0}^{n-1} c_e g^{3}_{n+1-e} + c_n g^{3}_1 - \frac{1}{2}\sum_{s=1}^{n} c_s c_{n+1-s} g^{3}_0 - e_{n+1} = \sum_{e=0}^{n} c_e g^{3}_{n+1-e} + \Big(-\frac{1}{2}\sum_{s=1}^{n} c_s c_{n+1-s} + e_{n+1}\Big)g^{3}_0$$

上式可写为

$$a^{3}_{n+1} = c_0 g^{3}_{n+1} + c_1 g^{3}_n + \cdots + c_n g^{3}_1 + c_{n+1} g^{3}_0 \tag{14}$$

其中,c_{n+1}满足关系式(11)$_{m=n+1}$. 因 c_j 与 e_j 是一一对应的,故(13)式所定义的解 a_{n+1}是方程(4)的通解. 事实上,利用(7)和(14)式以及(9)式有

$$f_{n+1} = \frac{1}{2}J\partial f_n - e a^{3}_{n+1} f_0 = \frac{1}{2}J\partial\sum_{s=0}^{n} c_s\bar{g}_{n-s} - \sum_{s=0}^{n+1} c_s g^{3}_{n+1-s}\bar{g}_0 =$$

$$\sum_{s=0}^{n} c_s\Big(\frac{1}{2}J\partial\bar{g}_{n-s} - g_0 g^{3}_{n+1-s}\Big) - c_{n+1} g^{3}_0\bar{g}_0 = \sum_{s=0}^{n} c_s\bar{g}_{n+1-s} + c_{n+1}\bar{g}_0$$

上式和(14)式证明了(10)当 $m=n+1$时成立. 即(10)、(11)式对任意 m成立. 定理证毕.

现假设

$$G_m = G_m(\lambda, a) = \sum_{j=0}^{m-1} a_j\lambda^{2(m-1-j)} \tag{15}$$

易证得 $(K-\lambda^2 J)G_m = Ka_{m-1} = Ja_m$

记 $V_m = \lambda e(SG_m)$. 定义孤子方程向量场为 $X_m = Jg_m$. 则由(15)式、(3)式和定理 2有 $(K-\lambda^2 J)G_n = c_0 X_m + c_1 X_{m-1} + \cdots + c_n X_0 \equiv \bar{X}_m$, $\partial V_m - [U, V_m] = \lambda e(S\bar{X}_m)$. 零曲率方程

$$U_{t_m} - V_{m,x} + [U, V_m] = \lambda e(T_{t_m} - S\bar{X}_m) = 0 \tag{16}$$

等价于

$$T_{t_m} = S\bar{X}_m = SJa_m = Ja_m \tag{17}$$

当 a_m取特解 g_m,即 $c_0 = 1, c_1 = \cdots = c_m = 0$时上式变为 $T_{t_m} = SX_m = Jg_m$,即

$$\frac{\partial}{\partial t_m}\begin{pmatrix} u \\ v \\ -\lambda \end{pmatrix} = Jg_m = \begin{pmatrix} 2g^2_m + 2vg^3_m \\ -2g^1_m - 2ug^3_m \\ 0 \end{pmatrix} = \begin{pmatrix} \partial g^1_{m-1} \\ \partial g^2_{m-1} \\ 0 \end{pmatrix} \tag{18}$$

它满足等谱条件 $\lambda_{t_m}=0$.

由 V_m 的定义和零曲率方程 (16)知,如定义

$$H_{t_m}=V_m H=\sum_{j=0}^{m-1} e(S a_j)\lambda^{2(m-j)-1}H \tag{19}$$

则方程 (1)和 (19)的解满足相容性条件 . U 和 V_m 是孤子方程族 (17)的一个 Lax对 .

由式 (9)和 (18)可递推计算 g_m 和 X_m. 求得显式孤子方程族 . 下面是应用 Mathematica[TM]数学软件计算出的部分结果,其中 u'表示 u_x, u''表示 u_{xx}, $u^{(n)}$表示$\frac{\partial^n u}{\partial^n x}$,等等 .

$$\begin{cases}\dfrac{\partial u}{\partial t_0}=u' \\[2mm] \dfrac{\partial v}{\partial t_0}=v'\end{cases},\quad \begin{cases}\dfrac{\partial u}{\partial t_1}=\dfrac{3}{2}u^2u'+\dfrac{1}{2}v^2u'+uvv'-\dfrac{1}{2}v'' \\[2mm] \dfrac{\partial v}{\partial t_1}=uvu'+\dfrac{1}{2}u^2v'+\dfrac{3}{2}v^2v'+\dfrac{1}{2}u''\end{cases}$$

$$\begin{cases}\dfrac{\partial u}{\partial t_2}=\dfrac{25}{8}u^4u'+\dfrac{15}{4}u^2v^2u'+\dfrac{5}{8}v^4u'+\dfrac{1}{4}vu'^2+\dfrac{5}{2}u^3vv'+\dfrac{5}{2}uv^3v'- \\ \qquad \dfrac{7}{4}uu'v'-\dfrac{3}{2}vv'^2+\dfrac{1}{4}uvu''-u^2v''-\dfrac{3}{4}v^2v''-\dfrac{1}{4}u^{(3)} \\[2mm] \dfrac{\partial v}{\partial t_2}=\dfrac{5}{2}u^3vu'+\dfrac{5}{2}uv^3u'+\dfrac{3}{2}uu'^2+\dfrac{5}{8}u^4v'+\dfrac{15}{4}u^2v^2v'+\dfrac{25}{8}v^4v'+ \\ \qquad \dfrac{7}{4}vu'v'-\dfrac{1}{4}uv'^2+\dfrac{3}{4}u^2u''-\dfrac{1}{4}uvv''+v^2u''-\dfrac{1}{4}v^{(3)}\end{cases}$$

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The Lax Representation of a Hierarchy of Soliton Equations

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Abstract For some eigenvalue problem, a map is defined to give the Lax pairs for a hierarchy of soliton equations. The inverse differential operator ∂^{-1} is not contained in the corresponding Lenard's equations $Ka_{j-1}=Ja_j$. The explicit general solutions of the Lenard's equations are linearly represented, by which the hierarchy of soliton equations is derived in explicit form.

Keywords soliton equation, Lax representation, symbolic operation

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