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共振非严格双曲守恒律组解的存在唯一性*

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摘　要　对共振非严格双曲守恒律方程组给出广义解的存在唯一性.
关键词　非严格双曲组，广义解，存在唯一性
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文 [1] 讨论了共振非线性双曲守恒律组，其 2×2形式为

$$\begin{cases} u_t+ f(u,a)_x= 0 & u(x,0)= u_0(x) \\ a_t= 0 & a(x,0)= a_0(x) \end{cases} \tag{1}$$

当 (1) 的特征根 $\lambda_1= f_u$ 和 $\lambda_2= 0$相等时，就会产生非严格双曲即共振现象.

文 [2, 3] 研究了式 (1) 等价形式弱解的存在性与渐近性质；文 [4] 讨论了式 (1) Glimm格式和 Godunov格式收敛速度的比较. 本文考察了式 (1) 的广义解存在唯一性.

设 $\prod_T= R\times [0,T],\ Q_{XT}= [-X,X]\times [0,T],\ T> 0,\ X> 0$

$$M_f= \sup\{|f|,|f_u|,|f_a|,|f_{uu}|,|f_{ua}|,|f_{aa}|\}$$

(u,a)称为式(1)在$\prod_{T_0}$的广义解是指成立

$$\int_0^T\int_R\{|u-q|\varphi_t+ \mathrm{sgn}(u-q)[f(u,a)-f(q,a)]\varphi_x-f(q,a)_x\,\mathrm{syn}(u-q)\varphi\}\,\mathrm{d}x\,\mathrm{d}t+$$
$$\int_R|u(x,0)-q|\varphi(x,0)\,\mathrm{d}x-\int_R|u(x,T)-q|\varphi(x,T)\,\mathrm{d}x\geqslant 0 \tag{2}$$

其中，$T_0> 0,\ T\in [0,T_0],\ \forall\ 0\leqslant\varphi\in C_0^\infty(\prod_{T_0}),\ \forall\ q\in[\inf u_0(x),\sup u_0(x)]$.

本文的主要结果是以下定理.

定理　设 $f(u,a)$有直到二阶的连续偏导数，$u_0(x)$，$a(x)$，$a'(x)$均属于 BV空间，则对 $\forall\ T> 0$，方程 (1) 在$\prod_T$存在广义解 $u(x,t)$，并有性质① $|u(x,t)|\leqslant M$; $\mathrm{TV}(u(\cdot,t))\leqslant(\mathrm{TV}(u_0)+C_0)e^{C_1t}$; ③ $\|u(\cdot,t)-u(\cdot,s)\|_{L^1}\leqslant K|t-s|$ 其中，$(x,t)\in\prod_T,\ s\in[0,T],\ M,K,C_0,C_1$是仅与 f，u_0,a,T有关的正常数，$\mathrm{TV}(u)$为 u的全变差.

若再有

$$(u_0(x),a(x))= (u_0^k,a^k),\text{当 } x\geqslant k;\ (u_0^L,a^L)\text{当 } x\leqslant L \tag{3}$$

其中，u_0^k，a^k，u_0^L，a^L为常数，$k> 0$，$L<0$为给定常数，而(u,a)，(v,a)为方程 (1) 对应于(u_0,a)，$(v_0,\bar a)$的广义解，其中，(u_0,a)，$(v_0,\bar a)$满足(3)式，则有

$$\|u(\cdot,t)-v(\cdot,t)\|_{L^1}\leqslant\left\{\|u_0-v_0\|_{L^1}+m_0\|a-\bar a\|_{L^1}+m_1\|a_x-\bar a_x\|_{L^1}\right\}e^{\bar m t}$$

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其中，$t\in[0,T]$，m_0，m_1，$\bar m$ 是仅与 f，u_0，v_0，a，$\bar a$，a_x，$\bar a_x$ 有关的正常数．

设 $u(x,t)$是下述抛物方程初值问题的古典解

$$u_t+f(u,a)_a-\varepsilon u_{xx}=0\quad u|_{t=0}=u_0(x)\tag{4}$$

引理 1 $|u(x,t)|\leqslant M$，$M=\max\{\|u_0\|e^t,M_f\|a_x\|\}$，$(x,t)\in\prod_T$ (5)

证明 设 $u=ve^t$，代入式 (4)，则有 $v_t+f_uv_x+v+f_aa_xe^{-t}-\varepsilon v_{xx}=0$

又设 $s(x,t)=\bar c\phi(x,t)+M-v(x,t)$，其中 $\phi=\exp\{(\lambda_1t+1)x^2+\lambda_2t\}$，$\lambda_1$，$\lambda_2$为待定常数，$\bar c=\delta\phi^{-1}_{(x_0,t_0)}$，$\delta$为任意给定正数，$(x_0,t_0)$为$\prod_T$中任意取定点．

设有算子 L: $L(p)=\frac{\partial p}{\partial t}+f_u\frac{\partial p}{\partial x}+p-\varepsilon p_{xx}$ （p 为光滑函数）

则有
$$\begin{aligned}L(s)&=\bar cL(\phi)+L(M)-L(v)\geqslant\\&\phi\{[\lambda_1-2(\lambda_1t+1)(2\varepsilon(\lambda_1t+1)+M_f)]x^2+\lambda_2+1-2(\lambda_1t+1)(\varepsilon+M_f)\}\end{aligned}$$

取 $\lambda_1\geqslant4(4\varepsilon+M_f)$，$0<t\leqslant\bar t=1/\lambda_1$，$\lambda_2\geqslant4(\varepsilon+M_f)-1$，则有 $L\phi\geqslant0$，再取 L充分大使得 $e^{-L^2}\|v(x,t)\|\leqslant\bar c$（$\|\cdot\|$ 表示 L^∞模，下同）．而当 $x=\pm L$ 时，$s(\pm L,t)\geqslant\bar c\phi(\pm L,t)-v(\pm L,t)+M\geqslant0$，$s(x,0)=\bar c\phi(x,0)+M-u_0(x)\geqslant0$，由最大值原理，对 $(x_0,t_0)\in Q_{L\bar t}$，成立

$$s(x_0,t_0)=\bar c\phi(x_0,t_0)+M-v(x_0,t_0)\geqslant0\quad 即\quad v(x_0,t_0)\leqslant M+\delta$$

若 $(x_0,t_0)\bar\in Q_{L\bar t}$，逐步设 $u=ve^{t-\bar t}$，…，$u=v^{t-n\bar t}$，有限步后可归为上述．同理有 $v(x_0,t_0)\geqslant M-\delta$．由正数 δ的任意性，即知引理 1得证．

引理 2 $$\mathrm{TV}(u(\cdot\ t))\leqslant(\mathrm{TV}(u_0)+C_0)e^{C_1t}\qquad t\in[0,T]\tag{6}$$
其中，$C_0=\mathrm{TV}(a)+\mathrm{TV}(a_x)/\|a_x\|$，$C_1=M_f\|a_x\|$．

证明 将式 (4) 两端关于 x 求导可得 $(u_x)_t+(f_uu_x)_x+(f_aa_x)_x-\varepsilon(u_x)_{xx}=0$

从而 $$(u_x-\theta)_t+[f_u(u_x-\theta)]_x+\theta(f_u)_x+(f_aa_x)_x-\varepsilon(u_x-\theta)_{xx}=0\tag{7}$$

其中，$\theta\in[0,1]$．设在 Q_{XT}上使 $u_x-\theta=0$的集合为 $E(\theta)$，则有 $\mathrm{mes}E(\theta)=0$ a.e于 $\theta\in[0,1]$．记 $E_t(\theta)$为 $E(\theta)$的 t水平集，则有 $[-X,X]\setminus E_t(\theta)=\sum(a_k,b_k)$．$(a_k,b_k)$彼此不交，且 $(u_x-\theta)|_{a_k}=(u_x-\theta)|_{b_k}=0$，$u_x-\theta$在 (a_k,b_k)保持定号．设非负 $\psi\in C_0^\infty(Q_{XT})$，以 $\psi\cdot\mathrm{sgn}(u_x-\theta)$乘以 (7) 式并在 Q_{XT}上积分，适当整理后令 $\theta\to0+$ 则有

$$\begin{aligned}&\int_{|x|\leqslant X}\{|u_x(x,T)|\cdot\psi(x,T)-|u(x,0)|\cdot\psi(x,0)\}dx\leqslant\\&\iint_{Q_{XT}}\{|u_x|\cdot\psi_t+[f_u|u_x|+f_aa_x\mathrm{sgn}u_x]\psi_x+\varepsilon|u_x|\psi_{xx}\}dx\,dt+\\&\int_0^T\Big(\sum_k|(f_aa_x\psi)(b_k)-(f_aa_x\psi)(a_k)|\Big)dt\end{aligned}\tag{8}$$

注意到上式最后项中的和式部分

$$\begin{aligned}&\sum_k|(f_aa_x\psi)(b_k)-(f_aa_x\psi)(a_k)|\leqslant M_f\{\|\psi\|\sum_k|a_x(b_k)-a_x(a_k)|+\|a_x\|\sum_k|(\psi(b_k)-\\&\psi(a_k)|\}+\|\psi\|\|a_x\|M_f\Big\{\sum_k|u(b_k)-u(a_k)|+\sum_k|a(b_k)-a(a_k)|\Big\}\leqslant\\&M_f\begin{Bmatrix}\|\psi\|\ \mathrm{TV}(a_x)+\\\|a_x\|\ \mathrm{TV}(\psi(\cdot\ t))\end{Bmatrix}+\|\psi\|\|a_x\|M_f\{\mathrm{TV}(u(\cdot\ t))+\mathrm{TV}(a)\}\end{aligned}\tag{9}$$

再取 $\psi(x,t)=W_\lambda(x)\zeta(x/N)$．其中，$W_\lambda(x)=e^{-\lambda\sqrt{1+x^2}}$，$\zeta(u)\in C_0^\infty$ 且 $\zeta(u)=\begin{cases}1 & |u|\leqslant1\\0 & |u|\geqslant2\end{cases}$，则当 $\lambda\to0+$，$N\to+\infty$时，$\psi\to1$，$\psi_x=O(1/N)$．将此 ψ代入 (8) 式并注意到式

(9)，令 $\lambda=1/N=1/X$，让 $X\to+\infty$ 即有

$$\mathrm{TV}(u(\cdot\ t))\leqslant \mathrm{H}(t)+C_1\int_0^t \mathrm{TV}(u(\cdot\ t))\,\mathrm{d}t$$

其中，$\mathrm{H}(t)=\mathrm{TV}(u_0)+M_f(\mathrm{TV}(a_x)+\|a\|\ \mathrm{TV}(a))t, C_1=M_f\|a_x\|$．由此即得（6）式．

引理 3　成立 $\int_R|u_t(x,t)|\,\mathrm{d}x\leqslant K,\forall t\in[0,T]$，正数 K 仅与 $f,u_0,a,T\ (T>0)$ 有关．

证明　由式 (3) 得 $(u_t)_t+(f_u u_t)_x-X(u_t)_{xx}=0$，应用引理 2中类似方法即可证得引理 3.

由引理 1-3，即可证明定理解的存在性．现证明解对初值的连续依赖性．

设 $\mathrm{T}=\dfrac{\mathrm{j}(x)}{C}$，而 $\mathrm{j}(x)=\begin{cases}\exp\{1/(|x|^2-1)\} & |x|<1\\ 0 & |x|\geqslant 1\end{cases}$，$C=\int_R \mathrm{j}(x)\,\mathrm{d}x$．又记 $\mathrm{T_X}(x)=\mathrm{T}(x/X)/X$，则易验证

① $\mathrm{T_X}(x)\geqslant 0,\ \forall x\in R$；② 存在与 X 无关正常数 M_T 有 $\mathrm{T_X}(x)\leqslant M_\mathrm{T}/X$，$|\mathrm{T}'_\mathrm{X}(x)|\leqslant M_\mathrm{T}/X^2$.

在（2）式中取 $q=v(y,\mathrm{f})$，$\mathrm{h}=\mathrm{h}(x,t;y,\mathrm{f})=\mathrm{T_X}(x-y)\mathrm{T_X}(t-\mathrm{f})$ 并在 $\prod_T$ 上对 y，f 积分(以下积分限自明)：

$$\iiiint\{|u(x,t)-v(y,\mathrm{f})|\mathrm{h}_t+\mathrm{sgn}(u(x,t)-v(y,\mathrm{f}))[f(u(x,t),a(x))-$$
$$f(v(y,\mathrm{f}),a(x))]\mathrm{h}_x-f(v(y,\mathrm{f}),a(x))_x\,\mathrm{sgn}(u(x,t)-v(y,\mathrm{f}))\mathrm{h}\}\,\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\mathrm{f}+$$
$$\iiint|u(x,0)-v(y,\mathrm{f})|\mathrm{h}(x,0;y,\mathrm{f})\,\mathrm{d}x\mathrm{d}y\mathrm{d}t-\iiint|u(x,T)-v(y,\mathrm{f})|\mathrm{h}(x,T;y,\mathrm{f})\,\mathrm{d}x\mathrm{d}y\mathrm{d}\mathrm{f}\geqslant 0$$

对 $v(y,\mathrm{f})$ 和 $\bar a(y)$ 也有相应的（2）式，以 $u(x,t)$ 代 q，前述 $\mathrm{h}(x,t;y,\mathrm{f})$ 代 h 并在 $\prod_T$ 上关于 x，t 积分有

$$\iiiint\{|v(y,\mathrm{f})-u(x,t)|\mathrm{h}_\mathrm{f}+\mathrm{sgn}(v(y,\mathrm{f})-u(x,t))[f(v(y,\mathrm{f}),a(y))-$$
$$f(u(x,t),a(y))]\mathrm{h}_y-f(u(x,t),\bar a(y))_y\,\mathrm{sgn}(v(y,\mathrm{f})-u(x,t))\mathrm{h}\}\,\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\mathrm{f}+$$
$$\iiint|v(y,0)-u(x,t)|\mathrm{h}(x,t;y,0)\,\mathrm{d}y\mathrm{d}x\mathrm{d}\mathrm{f}-\iiint|v(y,T)-u(x,t)|\mathrm{h}(x,t;y,T)\,\mathrm{d}y\mathrm{d}x\mathrm{d}\mathrm{f}\geqslant 0$$

上述两式相加并适当整理可得

$$\iiiint(u(x,t)-v(y,\mathrm{f})|(\mathrm{h}_t+\mathrm{h}_\mathrm{f})\,\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\mathrm{f}+$$
$$\iiiint\mathrm{sgn}(u(x,t)-v(y,\mathrm{f}))\{[f(u(x,t),a(x))-f(v(y,\mathrm{f}),a(x))]\mathrm{h}_x-$$
$$[f(v(y,\mathrm{f}),\bar a(y))-f(u(x,t),\bar a(y))]\mathrm{h}_y\}\,\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\mathrm{f}+$$
$$\iiiint\mathrm{sgn}(u(x,t)-v(y,\mathrm{f}))\{[f(u(x,t),\bar a(y))_y-f(v(y,\mathrm{f}),a(x))_x\}\mathrm{h}\,\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\mathrm{f}+$$
$$\iiint(u(x,0)-v(y,\mathrm{f})|\mathrm{h}(x,0;y,\mathrm{f})\,\mathrm{d}x\mathrm{d}y\mathrm{d}\mathrm{f}-\iiint u(x,T)-v(y,\mathrm{f})|\mathrm{h}(x,0;y,\mathrm{f})\,\mathrm{d}x\mathrm{d}y\mathrm{d}\mathrm{f}+$$
$$\iiint v(y,0)-u(x,t)|\mathrm{h}(x,t;y,0)\,\mathrm{d}x\mathrm{d}t\mathrm{d}y-\iiint v(y,T)-u(x,t)|\mathrm{h}(x,t;y,T)\,\mathrm{d}x\mathrm{d}t\mathrm{d}y\geqslant 0\quad(10)$$

按顺序以相应记号记之为 $I_1+I_2+I_3+J_1-J_2+J_3-J_4\geqslant 0$，由 $\mathrm{h}_t+\mathrm{h}_\mathrm{f}=0$ 可知 $I_1=0$.

设

$$I_3^0=\iiiint\mathrm{sgn}(u(x,t)-v(y,\mathrm{f}))\{f_a(u(x,t),\bar a(y))(\bar a_y-a_y)+$$
$$[f_a(u(x,t),\bar a(x))-f_a(u(x,t),a(x)]a_x+$$
$$[f_a(u(y,\mathrm{f}),a(x))-f_a(v(y,\mathrm{f}),a(x))]a_x\}\mathrm{h}\,\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\mathrm{f}$$

则 $|I_3^0|\leqslant M_f\iiiint\{|\bar a_y-a_y|+\|a_x\|\,|\bar a(x)-a(x)|+\|a_x\|\,|u(y,\mathrm{f})-v(y,\mathrm{f})|\}\mathrm{h}\,\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\mathrm{f}$

但 $$\iiiint |u(y,f)-v(y,f)|\, h\,dx\,dt\,dy\,df=\frac{1}{2}\int_0^T\int_R |u(y,f)-v(y,f)|\,dy\,df$$

$$\iiiint |\bar{a}_y-a_y|\,h\,dx\,dt\,dy\,df=\frac{T}{2}\int_R|\bar{a}_y-a_y|\,dy$$

$$\iiiint |\bar{a}(x)-a(x)|\,h\,dx\,dt\,dy\,df=\frac{T}{2}\int_R|\bar{a}(x)-a(x)|\,dx$$

所以 $$|I_3^0|\leqslant M_f\frac{1}{2}\{T\int_R|\bar{a}_y-a_y|\,dy+T\|a_x\|\int_R|\bar{a}(x)-a(x)|\,dx+\int_0^T\int_R|u(y,f)-v(y,f)|\,dy\,df\}.$$

设 $e(I_3)=I_3-I_3^0$，则有

$$e(I_3)=\iiiint \mathrm{sgn}\,(u(x,t)-v(y,f))\{f_a(u(x,t),\bar{a}(y))a_y-f_a(u(x,t),\bar{a}(y))a_x+[f(u(x,t),\bar{a}(y))-f_a(u(x,t),\bar{a}(x))]a_x+[f_a(u(x,t),a(x))-f_a(u(y,f),a(x))]a_x\}\,h\,dx\,dt\,dy\,df$$

所以 $$|e(I_3)|\leqslant M_f\iiiint\{|a_y-a_x|+\|a\|\,|\bar{a}(y)-\bar{a}(x)|+\|a_x\|\,|(u(x,t)-v(y,f))|\}\,h\,dx\,dt\,dy\,df$$

由(3)式可知

$$\iiiint (u(x,t)-v(y,f)|\,h\,dx\,dt\,dy\,df=\int_{-r}^{r}\int_0^T\int_{t-X}^{t+X}\int_{x-X}^{x+X}(u(x,t)-v(y,f)|\,h\,dx\,dt\,dy\,df\leqslant$$
$$M\int_{-r}^{r}\int_0^T\left\{\frac{1}{X^2}\int_{t-X}^{t+X}\int_{x-X}^{x+X}|(u(x,t)-v(y,f))|\,dy\,df\right\}dx\,dt$$

由 $u\in L^1_{loc}$ 故 $\prod_T$ 中的点 a.e 为 $u(x,t)$ 的 Lebesgue 点，而在 Q_{XT} 中上式 { } 部分 $\leqslant M|u(x,t)|$ （M 为正常数与 X 无关）由控制收敛定理即知当 $X\to 0+$ 时上式 $\to 0$. 同理 $\iiiint|\bar{a}(y)-a(y)|\,h\,dx\,dt\,dy\,df\to 0$, $\iiiint|\bar{a}(x)-a(x)|\,h\,dx\,dt\,dy\,df\to 0\ (X\to 0+)$.

由此有 $$I_3=I_3^0+(I_3-I_3^0)=I_3^0+e(I_3),\quad e(I_3)\to 0\ (X\to 0+)\tag{11}$$

设 $$J_1^0=\iiint (u(x,0)-v(x,0)|\,h(x,0,y,f)\,dx\,dy\,df$$

则 $$J_1^0=\int_0^T T_X(-f)\,df\int_R(|u(x,0)-v(x,0)|\circ\int_R T_X(x-y)\,dy)\,dx=\frac{1}{2}\int_R(|u(x,0)-v(x,0)|\,dx$$

令 $$e(J_1)=J_1-J_1^0$$

则 $$e(J_1)=\iiint\{|(u(x,0)-v(y,f)|-|u(x,0)-v(x,0)|\}\,h(x,0,y,f)\,dx\,dy\,df$$

故 $$|e(J_1)|\leqslant\iiint u(y,f)-v(x,0)|\,h(x,0,y,f)\,dx\,dy\,df$$

与证明 $e(I_1)\to 0$ 类似可得 $e(J_1)\to 0,\ (X\to 0+)$.

所以 $$J_1=J_1^0+e(J_1)\quad e(J_1)\to 0,\ X\to 0+\tag{12}$$

同理可得 $$J_i=J_i^0+e(J_i)\quad e(J_i)\to 0,\ X\to 0+\quad (i=2,3,4)\tag{13}$$

其中 $$J_2^0=\frac{1}{2}\int_R(|u(x,T)-v(x,T)|\,dx$$

$$J_3^0=\frac{1}{2}\int_R(|v(y,0)-u(y,0)|\,dy,\quad J_4^0=\frac{1}{2}\int_R(|v(y,T)-u(y,T)|\,dy$$

注意到 $\int_R [\omega_\varepsilon(x-y)]_x \mathrm{d}x = 0$

所以

$$I_2 = \iiiint \operatorname{sgn}(u(x,t) - v(y,\tau)) \circ \Big\{ [f(u(x,t),a(x)) - f(v(y,\tau),a(x))] + [f(v(y,\tau),\bar{a}(y)) - f(u(x,t),\bar{a}(y))] \Big\} \eta_x \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau - \iiiint \operatorname{sgn}(u(y,\tau) - v(y,\tau)) \Big\{ [f(u(y,\tau),a(y)) - f(v(y,\tau),a(y))] + [f(v(y,\tau),\bar{a}(y)) - f(u(y,\tau),\bar{a}(y))] \Big\} \eta_x \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau$$

从而

$$|I_2| \leqslant \iiiint \Big| \operatorname{sgn}(u(x,t) - v(y,\tau)) \{ [f(u(x,t),a(x)) - f(v(y,\tau),a(x))] + f(v(y,\tau),\bar{a}(y)) - f(u(x,t),\bar{a}(y))] \} - \operatorname{sgn}(u(y,\tau) - v(y,\tau)) \circ \{ [f(u(y,\tau),a(y)) - f(v(y,\tau),a(y)) + f(v(y,\tau),\bar{a}(y)) - f(u(y,\tau),\bar{a}(y))] \} \Big| |\eta_x| \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau$$

但上式中 $|\{\}| \leqslant M_f \{ |a(x) - \bar{a}(y)| |u(x,t) - v(y,\tau)| + |a(x) - a(y)| |u(y,\tau) - v(y,\tau)| \}$.

所以

$$|I_2| \leqslant M_f \iiiint \{ |a(x) - \bar{a}(y)| |u(x,t) - v(y,\tau)| + |a(x) - a(y)| |u(y,\tau) - v(y,\tau)| \} |\eta_x| \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau = M_f (I_2^{(1)} + I_2^{(2)})$$

而

$$I_2^{(2)} \leqslant \iiiint \|a_x\| |x-y| |u(y,\tau) - v(y,\tau)| \omega_\varepsilon(x-y) \omega_\varepsilon(t-\tau) \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau \leqslant \|a_x\| M_T \int_0^T \int_R |u(y,\tau) - v(y,\tau)| \mathrm{d}y\mathrm{d}\tau$$

$$I_2^{(1)} \leqslant \iiiint \Big\{ |a(x) - a(y)| + |a(y) - \bar{a}(y)| \Big\} |u(x,t) - u(y,\tau)| |\eta_x| \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau$$

但

$$\iiiint \{ |a(x) - a(y)| |u(x,t) - u(y,\tau)| |\eta_x| \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau \leqslant \|a_x\| \iiiint |x-y| |u(x,t) - u(y,\tau)| |\eta_x| \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau \leqslant M_T^2 \|a_x\| \int_{-\infty}^{\infty} \int_0^T \left(\frac{1}{\varepsilon^2} \int_{t-\varepsilon}^{t+\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} |u(x,t) - u(y,\tau)| \mathrm{d}y\mathrm{d}\tau \right) \mathrm{d}x\mathrm{d}t \xrightarrow{\varepsilon \to 0+} 0$$

又

$$\iiiint |a(y) - \bar{a}(y)| |u(x,t) - u(y,\tau)| |\eta_x| \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau \leqslant \|a(y) - \bar{a}(y)\| \circ \Big\{ \iiiint |u(x,t) - u(y,t)| |\eta_x| \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau + \iiiint |u(y,t) - u(y,\tau)| |\eta_x| \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau \Big\} = \|a(y) - \bar{a}(y)\| \circ (R_1 + R_2)$$

$$R_1 = \int_0^T \left\{ \int_R \left(\frac{u(x,t) - u(y,t)}{|x-y|} |x-y| \circ \int_R |\omega'_\varepsilon(x-y)| \mathrm{d}y \right) \mathrm{d}x \circ \int_0^T \omega_\varepsilon(t-\tau) \mathrm{d}\tau \right\} \mathrm{d}t \leqslant \frac{1}{2} M_T \int_0^T \mathrm{TV}(u(\circ\ t)) \mathrm{d}t \leqslant \frac{1}{2} M_T (\mathrm{TV}(u_0) + C_0) \mathrm{e}^{c_1 T}$$

$$R_2 = \int_0^T \left\{ \int_R \left(\frac{u(y,t) - u(y,t)}{|t-\tau|} |t-\tau| \circ \int_R |\omega'_\varepsilon(x-y)| \mathrm{d}x \right) \mathrm{d}y \circ \int_0^T \omega_\varepsilon(t-\tau) \mathrm{d}\tau \right\} \mathrm{d}t \leqslant \frac{1}{2} M_T \int_0^T \frac{\| u(\circ\ t) - u(\circ\ \tau) \|_{L^1}}{|t-\tau|} \mathrm{d}t \leqslant \frac{1}{2} M_T K \circ T$$

从而

$$I_2^{(1)} \leqslant \|a(y\} - \bar{a}(y)\| \circ \frac{M_T}{2} \Big\{ \Big(\mathrm{TV}(u_0) + C_0 \Big) \mathrm{e}^{c_1 T} + KT \Big\}$$

故有

$$|I_2| \leqslant M_f \frac{M_T}{2} \|a(y) - \bar{a}(y)\| \circ \Big\{ \Big(\mathrm{TV}(u_0) + C_0 \Big) \mathrm{e}^{c_1 T} + KT \Big\} + M_T \|a_x\| \int_0^T \|u(\circ\ \tau) - u(\circ\ \tau)\|_{L^1} \mathrm{d}\tau \quad (14)$$

将 $I_1=0$及（11-14）式代入式(10)并适当整理即有

$$\int_R |u(x,t)-v(x,t)|\,dx \leqslant \int_R |u(x,0)-v(x,0)|\,dx + M_f\left\{\|a_x-\bar{a}_x\|_{L^1}\left[\frac{M_T}{2}[\mathrm{TV}(u_0)+C_0)e^{c_1T}+KT]+\frac{T}{2}\right\}+\|a_x\|\|a_x-\bar{a}_x\|_{L^1}+\left(\frac{M_f}{2}+M_T\|a_x\|\right)\int_0^T\|u(\cdot f)-u(\cdot f)\|_{L^1}\,df$$

即

$$\|u(\cdot t)-v(\cdot t)\|_{L^1}\leqslant\|u_0-v_0\|_{L^1}+m_0\|a_x-\bar{a}_x\|_{L^1}+m_1\|a_x-\bar{a}_x\|_{L^1}+\bar{m}\int_0^t\|u(\cdot f)-u(\cdot f)\|_{L^1}\,df \qquad t\in[0,T]$$

其中，$m_0=\frac{M_f}{2}\left\{M_T[\mathrm{TV}(u_0)+C_0)e^{c_1T}+\mathrm{TV}(u_0)+\mathrm{TV}(a))\cdot T]+T\right\}$

$m_1=\frac{M_f}{2}\|a_x\|$，$\bar{m}=\frac{M_f}{2}+M_T\|a_x\|$，由 Gronwall不等式即证得定理的第 2部分.

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Existence and Uniqueness of the Solution for A Resonant Non-strictly Hyperbolic System

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Abstract The existence and uniqueness of the generalized solution for a resonant non-strictly hyperbolic system are established.

Keywords nonstrictly hyperbolic systems, generalized solution, existence and uniqueness

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