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Dirichlet Problem of Bi-analytic Functions of Four Variables *

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Abstract：Dirichlet problem of bi-analytic functions of four variables on the bounded simply-connected domain with smooth boundary is studied. The results, based on quaternionic analysis theory, are of both theoretical and practical interest.

Keywords：boundary value problem；Bergmann kernel；quaternionic analysis

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A function theory which is called（λ, k）-bianalytic function theory has been developed by Hua，et al[1]. Bi-analytic functions of several variables have been studied by Begehr and Kumar[2]. In [3]，Kumar considers the generalized Riemann boundary value problem by introducing Cauchy integral of bi-analytic type in two variables, In this paper, we obtain representation of solution of（1, 1, λ, λ）-analytic type in four variables on bounded simply connected domain Ω with smooth boundary in $\mathbf{C}^2$ by applying theory of hyperholomorphic quaternionic analysis.

With respect to practical and theoretical interest of bi-analytic, we consider the following bi-analytic function:

$$\begin{cases} \dfrac{\partial u_k}{\partial x_1} + (-1)^{4-k} \dfrac{\partial u_{5-k}}{\partial x_2} = \Theta_1^{(k,\ 5-k)} \\ \dfrac{\partial u_k}{\partial x_3} + (-1)^{4-k} \dfrac{\partial u_{5-k}}{\partial x_4} = 0 \end{cases} \tag{1}$$

for $1 \leqslant k \leqslant 4$

and
$$\frac{\partial \Theta_1^{(m,\ 5-m)}}{\partial x_{2l-1}} + (-1)^{l-1} \lambda \frac{\partial \Theta_1^{(5-m,\ m)}}{\partial x_{2l}} = 0, \text{ for } m = 1,\ 3 \tag{2}$$

$$-\lambda \frac{\partial \Theta_1^{(m,\ 5-m)}}{\partial x_{2l-1}} + (-1)^{l-1} \frac{\partial \Theta_1^{(5-m,\ m)}}{\partial x_{2l}} = 0, \text{ for } m = 2, 4 \tag{3}$$

for any $l = 1,\ 2$, where $\lambda \neq 0,\ 1,\ -1$

A combination of the above identities shows that

$$\begin{cases} \dfrac{\partial f^k}{\partial \bar{z}_1} = \dfrac{\lambda - 1}{4\lambda} \phi_1^k(z) + \dfrac{\lambda + 1}{4\lambda} \overline{\phi_1^k(z)} \\ \dfrac{\partial f^k}{\partial \bar{z}_2} = 0, \dfrac{\partial \phi_1^k}{\partial \bar{z}_1} = 0, \dfrac{\partial \phi_1^k}{\partial \bar{z}_2} = 0 \end{cases} \tag{4}$$

where　$f^k = u_k + \mathrm{i} u_{5-k}$，$z_i = x_{2j-1} + \mathrm{i} x_{2j}$　and

$\phi_1^k = \Theta_1^{(k, 5-k)} - \mathrm{i}\lambda \Theta_1^{(5-k, k)}$

The rest of the paper is organized as follows： in section 1 introduce theory of hyperholomorhpic quater-

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nionic analysis and in section 2 discuss Dirichlet problem.

1 Elements of Hyperholomorphic Quaternionic Analysis

We denote by H as usual the set of real quaternions, which means that if $q \in H$, then

$$q = \sum_{k=0}^{3} q_k \circ i_k$$

where $i_0 i_k = i_k i_0 = i_k,\ i_k^2 = - i_0$ for $k \in \{1, 2, 3\} := N_3$

$i_1 i_2 = - i_2 i_1 = i_3,\ i_2 i_3 = - i_3 i_2 = i_1$

$i_3 i_1 = - i_1 i_3 = i_2;\ \{q_k\} \subset \mathbf{R}$

Let $\psi := \{\psi^0, \psi^1, \psi^2, \psi^3\}$ be an orthonormal (in the usual Euclidean sense) system of quaternions sometimes we call ψ a structural set.

On the set $C^1(\Omega, H)$, define the operators ${}^{\psi}D$ and D^{ψ} by the equalities

$${}^{\psi}D[f] := \sum_{k=0}^{3} \psi^k \frac{\partial f}{\partial x_k} =: \sum_{k=0}^{3} \psi^k \partial_k[f]\ ;\ D^{\psi}[f] := \sum_{k=0}^{3} \frac{\partial f}{\partial x_k} \psi^k =: \sum_{k=0}^{3} \partial_k[f]\ \psi^k.$$

Let Δ be the four-dimensional Laplace operator:

$\Delta := \sum_{k=0}^{3} \partial_k^2$. Define Δ on H-valued functions by H-valued functions by $\Delta[f] := \sum_{k=0}^{3} \Delta[f_k] \circ i_k$, then on $C^2(\Omega, H)$ the following equalities hold:

$$\Delta = {}^{\psi}D\,{}^{\bar\psi}D = {}^{\bar\psi}D\,{}^{\psi}D = D^{\psi} D^{\bar\psi} = D^{\bar\psi} D^{\psi}$$

Fix a structural set ψ and a domain Ω.

We introduce the sets[4, 5]:

$${}^{\psi}m(\Omega) := \ker {}^{\psi}D = \{f \in C^1(\Omega, H) \mid {}^{\psi}D[f] = O_\Omega\},$$

$$m^{\psi}(\Omega) := \ker D^{\psi} = \{f \in C^1(\Omega, H) \mid D^{\psi}[f] = O_\Omega\}.$$

The following fundamental formulae are widely known and may be found in many sources.

The quaternionic Stokes formula: Let Ω be a bounded domain in $\mathbf{R}^4$ with a smooth boundary $\Gamma = \partial\Omega$, $\{f, g\} \subset C^1(\bar\Omega; H)$, then

$$\int_{\partial\Omega} g \sigma_{\psi_x} f = \int_{\Omega} (D^{\psi}[g] f + g\,{}^{\psi}D[f])\mathrm{d}x \tag{5}$$

Borel-Pompeiu quaternionic formula: Let Ω be a bounded domain in $\mathbf{R}^4$ with a smooth boundary $\Gamma = \partial\Omega$, $f \in C^1(\bar\Omega; H)$, then

$$f(x) = \int_{\partial\Omega} K_{\psi}(\tau - x)\sigma_{\psi_\tau} f(\tau) - \int_{\Omega} K_{\psi}(\tau - x)\,{}^{\psi}D[f(\tau)]\,\mathrm{d}\tau \tag{6}$$

Cauchy integral formula: Let Ω be a bounded domain in $\mathbf{R}^4$ with a smooth boundary $\Gamma = \partial\Omega$, $f \in {}^{\psi}m(\Omega; H) C^1(\bar\Omega; H)$, then

$$\int_{\partial\Omega} K_{\psi}(\tau - x)\sigma_{\psi_\tau} f(\tau) = \begin{cases} f(x),\ x \in \Omega, \\ 0,\ x \in \mathbf{R}^4 \setminus \bar\Omega \end{cases} \tag{7}$$

Let Ω be a bounded simply connected domain in $\mathbf{R}^4$ with a smooth boundary $\Gamma = \partial\Omega$, denote by g the harmonic (or classical) Green function of a domain Ω.

It is known (see, for instance, [6]) that such a function g uniquely exists. For an arbitrary structural set ψ, introduce the function ${}_{\psi}B(x, \zeta)$:

$$\psi B(x,\zeta) := D_\zeta^{\psi\psi} D_x[g](x,\zeta) = {}^{\psi}D_x {}^{\psi}D_\zeta[g](x,\zeta)$$

where $(x,\zeta)\in\Omega\times\Omega\setminus$diag. We shall call it the Bergmann left- ψhyperholomorphic kernel function and write it as the Bergmann h. h. kernel.

The Bergmann h. h. kernel has the following properties:

1) for any fixed $\zeta\in\Omega$, $\psi B(\circ,\zeta)\in{}^{\psi}M(\Omega)$

2) for any fixed $\zeta\in\Omega$, $\psi B(x,\circ)\in M^{\psi}(\Omega)$

3) $\psi B(x,\zeta)=\overline{\psi B(\zeta,x)}$ (8)

4) Reproducing property of Bergmann h. h. kernel. Define

$$^{\psi}T[f](x) := -\int_\Omega K_\psi(\tau-x)f(\tau)\mathrm{d}\tau = -\frac{1}{2\pi^2}\int_\Omega\frac{\sum_{k=0}^{3}\overline{\psi^k}(\tau_k-x_k)}{|\tau-x|^4}f(\tau)\mathrm{d}\tau$$

In the spaces $L_p(\Omega,H)$, $p>1$, the operator ${}^{\psi}D\,{}^{\psi}T$ is well-defined and its space

$$^{\psi}D\,{}^{\psi}T = I \tag{9}$$

Given structural set ψ, $p>1$, and domain Ω; denote by $\psi A^p(\Omega,H)$ the subset of $L_p(\Omega,H)$ consisting of all left- ψ-hyperholomorphic functions:

$$\psi A^p(\Omega,H) := L_p(\Omega,H)\cap{}^{\psi}M(\Omega,H)$$

we shall call $\psi A^p(\Omega,H)$ the Bergmann h. h. space. It has the following properties:

1) the Bergmann h. h. operator is self—adjoint on $L_2(\Omega,H)$

2) for any Ω, ψ, p as above, ${}^{\psi}B(L_p(\Omega,H))=\psi A^p(\Omega,H)$ and ${}^{\psi}B^2={}^{\psi}B$

Remark: Reformulations in the complex terms of (5) ~ (9) are obvious. For details see [5].

2 Dirichlet Problem

Let Ω be a simply connected bounded domain with smooth boundary in C^2, system (4) has a pair of solutions, such that

$$f^k(z_1,z_2)\,|_{\partial\Omega} = h_k(\zeta_1,\zeta_2) \tag{10}$$

where $h_k(\zeta_1,\zeta_2)$, $k=1,2$ are functions given on boundary $\partial\Omega$, and $\partial h_k/\partial\zeta_j$, $k=1,2$; $j=1,2$ are Hölder continuous on the boundary.

If $\psi := \{\psi^0,\psi^1,\psi^2,\psi^3\} := \{i_0,i_1,i_2,i_3\}\cong\{1,i,j,ji\}$, thenDirichlet problem may be written as

$$\begin{cases} {}^{\psi}Df^k = 2\left(\frac{\lambda-1}{4\lambda}\phi(z)_1^k+\frac{\lambda+1}{4\lambda}\overline{\phi(z)_1^k}\right) := \Psi^k(z) \\ \frac{\partial\phi_1^k}{\partial\bar z_1}=0,\qquad \frac{\partial\phi_1^k}{\partial\bar z_2}=0 \\ f^k(z_1,z_2)\,|_{\partial\Omega} = h_k(\zeta_1,\zeta_2) \end{cases} \tag{11}$$

Using the quaternionic Stokes formula (5)

$$f^k(z) = -\frac{1}{4}\int_\Omega D^{\psi}g(\zeta,z)\Psi^k\mathrm{d}\zeta\Lambda\mathrm{d}\bar\zeta + \int_{\partial\Omega}D_\zeta^{\psi}g(\zeta,z)\sigma_{\psi\zeta}f^k \tag{12}$$

We have $\frac{1}{4}\int_\Omega \psi B(z,\zeta)\Psi^k\mathrm{d}\zeta\Lambda\mathrm{d}\bar\zeta = \int_{\partial\Omega}\psi B(z,\zeta)\sigma_{\psi\zeta}h^k$ (13)

According to reproducing property of Bergmann h. h. kernel, (13) may be written as

$$\phi_1^k = \frac{1-\lambda}{1+\lambda}\frac{1}{4}\int_\Omega \overline{{}^{\psi}B(z,\zeta)\phi_1^k \mathrm{d}\zeta\Lambda\mathrm{d}\xi} + \overline{2\int_{\partial\Omega} {}^{\psi}B(z,\zeta)\sigma_{\psi\zeta}h^k}$$

It is known that the parameter $\lambda \in (0, 1]$, denoting $T\phi_1^k = \frac{1-\lambda}{1+\lambda}\frac{1}{4}\int_\Omega \overline{{}^{\psi}B(z, \zeta)\phi_1^k \mathrm{d}\zeta\Lambda\mathrm{d}\zeta}$.

With respect to the Bergmann h. h., opterator is self-adjoint on $L_2(\Omega, H)$, the operator ${}^{\psi}B$ is projector of $L_p(\Omega, H)$ onto ${}^{\psi}A^p(\Omega, H)$ and $\frac{\lambda-1}{\lambda+1} < 1$. According to the contractive mapping principle, we know that $(I-T)^{-1}$ exists, which allows us to express

$$\phi_1^k = (I-T)^{-1}\overline{2\int_{\partial\Omega} {}^{\psi}B(z,\zeta)\sigma_{\psi\zeta}h^k}$$

In fact, from (12) we get

$$\begin{aligned} f^k(z) = &-\frac{\lambda-1}{4}\int_\Omega D^{\psi}g(\zeta, z)(I-T)^{-1}\overline{\int_{\partial\Omega} {}^{\psi}B(\zeta, t)\sigma_{\psi t}h^k \mathrm{d}\zeta\Lambda\mathrm{d}\zeta} - \\ &\frac{\lambda+1}{4}\int_\Omega D^{\psi}g(\zeta, z)\overline{(I-T)^{-1}}\int_{\partial\Omega} {}^{\psi}B(\zeta, t)\sigma_{\psi t}h^k \mathrm{d}\zeta\Lambda\mathrm{d}\zeta + \int_{\partial\Omega} D^{\psi}_{\zeta}g(\zeta, z)\sigma_{\psi\zeta}h^k \end{aligned} \tag{14}$$

Since $h_k(\zeta_1, \zeta_2)$, $k=1, 2$ are functions given on boundary $\partial\Omega$, $\frac{\partial h_k}{\partial\zeta_j}$, $k=1, 2; j=1, 2$ are Hölder continuous on the boundary, when $z \to \partial\Omega$, the right side of equation (14) makes sense and f^k is uniquely determined by h^k.

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4 个变量的双解析函数的 Dirichlet 问题

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摘　要：讨论了 4 个变量的双解析函数 在有界单连通区域上的 Dirichlet 问题，建立在四元数分析基础上的结果有理论和实践的意义.

关键词：边值问题；Bergmann 核；四元数分析

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