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Asymmetry of Thurston's Pseudometric*

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Abstract：It is proved that the length spectrum metric is not quasiisometric to Thurston's pseudometric, and that Thurston's pseudometric is asymmetric in Teichmüller space. The range of asymmetry of Thurston's pseudometric is also described.

Keyworks：length spectrum；extremal length；Teichmüller metric

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Let S be a compact Riemann surface with genus g, $g > 1$. For any compact Riemann surface S_0 and any orientation preserved homeomorphism $f: S \to S_0$, we call the pair (S_0, f) a marked Riemann surface. Two marked Riemann surfaces (S_1, f_1) and (S_2, f_2) are equivalent if there is a conformal mapping $\phi: S_1 \to S_2$ which is homotopic to $f_2 \circ f_1^{-1}$. Denote $[S, f]$ to be the equivalent class of (S, f). The Teichmüller space $T(S)$ (also denoted by T_g) is the set of the equivalent classes $[S, f]$.

As we know, Teichmüller gave a complete metric on $T(S)$ using the dilatation of extremal quasiconformal mapping

$$d_T([S_1, f_1], [S_2, f_2]) = \ln K[f_0]$$

where $f_0: S_1 \to S_2$ is the extremal quasiconformal mapping in the homotopic class of $f_2 \circ f_1^{-1}$, and $K[f_0]$ is its dilatation.

Let $\Sigma_S = \{\gamma_i\}$ be the set of the representations of elements (not including the unit element) of the fundamental group $\pi_1(S)$ of surface S. Let $l_S(\gamma)$ be the shortest length under the Poincaré metric in the freely homotopic class of closed curve γ on the Riemann surface S. The sequence $\{l_S(\gamma_j)\}$ corresponding to Σ_S is called the length spectrum of S[1]. We introduce a metric on $T(S)$ using length spectrum of Riemann surface[2~4]

$$d([S_1, f_1], [S_2, f_2]) = \ln \rho([S_1, f_1], [S_2, f_2])$$

where $$\rho([S_1, f_1], [S_2, f_2]) = \sup_{\gamma \in \Sigma_{S_1}} \left\{ \frac{l_{S_2}(f(\gamma))}{l_{S_1}(\gamma)}, \frac{l_{S_1}(\gamma)}{l_{S_2}(f(\gamma))} \right\}$$

and $f = f_2 \circ f_0^{-1}$. This metric is called the length spectrum metric of the Teichmüller space $T(S)$. The Thurston's pseudometrics d_{p1} and d_{p2} are also defined by the length spectrum of Riemann surface as following.

$$d_{p_1}([S_1, f_1], [S_2, f_2]) = \ln \sup_{\alpha \in \Sigma_{S_1}} \frac{l_{S_2}(f(\alpha))}{l_{S_1}(\alpha)}, \quad d_{p_2}([S_1, f_1], [S_2, f_2]) = \ln \sup_{\alpha \in \Sigma_{S_1}} \frac{l_{S_1}(f(\alpha))}{l_{S_2}(\alpha)}$$

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where $f = f_2 \circ f_0^{-1}$. We know from the definition that

$$d_{p_1}([S_1, f_1], [S_2, f_2]) = d_{p_2}([S_1, f_1], [S_2, f_2]).$$

Remark1 Actually d_{p_1} and d_{p_2} are not strictly pseudometrics. For the terminology "pseudometric" has the following standard meaning: the non-negative function ρ in $X \times X$ defines a pseudometric on X if it satisfies:

$$\rho(x, x) = 0, \rho(x, y) = \rho(y, x)$$

and $$\rho(x, z) \leqslant \rho(x, y) + \rho(y, z)$$

for all x, y and z. (but we may have $\rho(x, y) = 0$ and $x \neq y$). For the sake of convenience, d_{p_1} and d_{p_2} are still called pseudometrics.

The study of the relations of various metrics or pseudometrics in $T(S)$ is very interesting. The length spectrum metric d is topologically equivalent to the Teichmüller metric d_T[5]. This result can also be obtained directly from the Thurston's embedding of Teichmüller space. The length spectrum metric is a complete metric. Furthermore we know that the metrics d, d_T are pseudometrics d_{p_1} and d_{p_2} are topologically equivalent to each other[3,4]. It is matural for us to consider the following problem: whether the metrics d, d_T and the pseudometrics d_{p_1}, d_{p_2} are quasiisometric to each other? In this paper, we prove that d_{p_1} is not quasiisometric to d_{p_2}, this result implies that the length spectrum metric d is not quasiisometric to the Thurston's pseudometrics d_{p_i}, $i = 1, 2$.

The problem whether the metric d is quasiisometric to d_{p_i}, $i = 1, 2$, or not, is equivalent to the following: whether there is constant $K \geqslant 1$ such that for any $\tau_1, \tau_2 \in T(S)$, the following inequality holds.

$$\frac{1}{K} d_{p_i}(\tau_1, \tau_2) \leqslant d(\tau_1, \tau_2) \leqslant K d_{p_i}(\tau_1, \tau_2), \ i = 1, 2.$$

We know from the definition that for any

$$\tau_1, \tau_2 \in T(S), d_{p_i}(\tau_1, \tau_2) \leqslant d(\tau_1, \tau_2), i = 1, 2$$

Then the above problem is equivalent to the following: whether there exists constant $K \geqslant 1$, such that for any $\tau_1, \tau_2 \in T(S)$,

$$d(\tau_1, \tau_2) \leqslant K d_{p_i}(\tau_1, \tau_2), i = 1, 2 \tag{1}$$

To solve the above problem, we first study whether the pseudometric d_{p_1} is quasiisometric to the pseudometric d_{p_2}. That is, whether there exists constant $K \geqslant 1$, such that for any $\tau_1, \tau_2 \in T(S)$,

$$\frac{1}{K} d_{p_1}(\tau_1, \tau_2) \leqslant d_{p_2}(\tau_1, \tau_2) \leqslant K d_{p_1}(\tau_1, \tau_2) \tag{2}$$

The answer to this problem is negative.

Theorem The pseudometric d_{p_1} is not quasiisometric to the pseudometric d_{p_2}.

Proof Suppose that there is a constant $K > 0$, such that

$$d_{p_2}(\tau_1, \tau_2) \leqslant K d_{p_1}(\tau_1, \tau_2)$$

for any $\tau_1, \tau_2 \in T(S)$.

Next we'll give an example to show that the above assumption is false. Let $\tau_0 = [S_0, f_0] \in T(S)$ and γ be a simple closed curve of S. Shrink the Poincaré length of $f_0(\gamma)$ to ε_n, where ε_n is a sequence of positive numbers which tends to zero. We get a sequence of points $\tau_n = [S_n, f_n]$, $n = 1, 2, \cdots$, in $T(S)$, corresponding to the above shrinking process.

For any closed curve $\alpha\in\Sigma_S$, α is not homotopic to γ, we have[7]

$$l_{\tau_n}(f_n(\alpha))\simeq C_1(\alpha)+C_2(\alpha)\,|\ln\varepsilon_n|\,i(\alpha,\gamma)$$

where $C_1(\alpha)$ and $C_2(\alpha)$ are constants depending on α and $i(\alpha,\gamma)$ is the geometric intersection of α and γ.

Without loss of generality, we take $\varepsilon_n=1/n$. From the definition

$$d_{p_1}(\tau_0,\tau_n)=\ln\sup_{\alpha\in\Sigma_S}\frac{l_{\tau_n}(f_n(\alpha))}{l_{\tau_0}(f_0(\alpha))}\simeq\ln\max\left\{\frac{1/n}{l_{\tau_0}(f_0(\gamma))},\ \sup_{\alpha'\in\Sigma'_S}\frac{C_1(\alpha')+C_2(\alpha')i(\gamma,\alpha')\ln n}{l_{\tau_0}(f_0(\alpha'))}\right\}$$

where $\Sigma'_S=\{\alpha: \alpha\in\Sigma_S, \alpha$ is not homotopic to $\gamma.\}$. Without loss of generality, we assume that $nl_{\tau_0}(f_0(\gamma))\geqslant 1$. Because for any $\alpha'\in\Sigma'_S$, $\dfrac{l_{\tau_n}(f_n(\alpha'))}{l_{\tau_0}(f_0(\alpha'))}\geqslant 1$

Then $$d_{p_1}(\tau_0,\tau_n)=\ln\sup_{\alpha'\in\Sigma_S}\frac{l_{\tau_n}(f_n(\alpha'))}{l_{\tau_0}(f_0(\alpha'))}\simeq\ln\sup_{\alpha'\in\Sigma'_S}\frac{C_1(\alpha')+C_2(\alpha')i(\gamma,\alpha')\ln n}{l_{\tau_0}(f_0(\alpha'))}$$

On the other hand, we have

$$d_{p_2}(\tau_0,\tau_n)=\ln\sup_{\alpha\in\Sigma_S}\frac{l_{\tau_0}(f_0(\alpha'))}{l_{\tau_n}(f_n(\alpha'))}=\ln(nl_{\tau_0}(f_0(\gamma)))$$

Denote $$M_n=\frac{d_{p_1}(\tau_0,\tau_n)}{d_{p_2}(\tau_0,\tau_n)}=\frac{\ln\sup_{\alpha\in\Sigma_S}[l_{\tau_n}(f_n(\alpha'))/l_{\tau_0}(f_0(\alpha'))]}{\ln(nl_{\tau_0}(f_0(\gamma)))}$$

Let $$M_n(\alpha)=\frac{\ln[l_{\tau_n}(f_n(\alpha'))/l_{\tau_0}(f_0(\alpha'))]}{\ln(nl_{\tau_0}(f_0(\gamma)))},$$

where $\alpha\in\Sigma_S$.

If α homotopic to γ, $M_n(\alpha)=-1$. If $\alpha\in\Sigma'_S$, then

$$M_n(\alpha)=\frac{O\left(\ln\dfrac{C_1(\alpha)+C_2(\alpha)i(\alpha,\gamma)\ln n}{l_{\tau_0}(f_0(\alpha))}\right)}{\ln(nl_{\tau_0}(f_0(\alpha)))}=$$

$$O\left(\frac{\ln\dfrac{C_1(\alpha)+C_2(\alpha)i(\alpha,\gamma)\ln n}{l_{\tau_0}(f_0(\alpha))}}{\ln(nl_{\tau_0}(f_0(\alpha)))}\right)=O\left(\frac{\ln\ln n}{\ln n}\right).$$

Therefore, for any

$$\alpha\in\Sigma'_S,\ \lim_{n\to\infty}M_n(\alpha)=0\tag{3}$$

By assumption, $M_n\geqslant 1/K$, becuase $M_n=\sup_{\alpha\in\Sigma_{f_n(S)}}M_n(\alpha)$, for any $\varepsilon>0$, there is $\alpha'\in\Sigma'_S$, such that $M_n(\alpha')>1/K-\varepsilon>0$. This contradicts(3). So the assumption that for any $\tau_1,\tau_2\in T(S)$, $d_{p_2}(\tau_1,\tau_2)\leqslant Kd_{p_1}(\tau_1,\tau_2)$ is false.

Because $d_{p_1}(\tau_1,\tau_2)=d_{p_2}(\tau_1,\tau_2)$

there is no constant K, such that for any

$$\tau_1,\tau_2\in T(S),\ d_{p_1}(\tau_1,\tau_2)\leqslant Kd_{p_2}(\tau_1,\tau_2)$$

Then (2) is not true. This completes the proof of the theorem.

Remark 2 In some sense, the theorem shows the extend of the lack of symmetry of the pseudometrics d_{p_i}, $i=1,2$.

From the definition of d and above results, we know that (1) is not true.

Corollary The length spectrum metric d is not quasiisometric to the Thurston's pseudometrics d_{p_i}, $i = 1, 2$.

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Thurston 拟度量的非对称性

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摘　要: 证明了在 Teichmüller 空间中, 长度谱度量不拟等距于 Thurston 拟度量, 证明了 Thurston 拟度量的非对称性, 刻划了 Thurston 拟度量不对称的程度.

关键词: 长度谱; 极值长度; Teichmüller 度量

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