$U(1)$ -equivariant Seiberg-Witten-Floer Homology for Homology 3-Sphere*

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Abstract: Appling the Morse-Bott theory to Chern-Simons functional the $U(1)$ -equivariant Seiberg-Witten-Floer homology and cohomology of Z -homology 3-sphere, their pairing and module structures are defined, and it is proved that they are topological invariants.

Keywords: $U(1)$ -equivariant; Seiberg-Witten-Floer cohomology and homology; topological invariants

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Seiberg and Witten[1] discovered their monopole equations in 1994. Kronheimer and Mrowka[2] soon realized its importance and applied it to the long-standing Thom conjecture. Taubes proved that symplectic 4-manifolds have nontrivial invariants and settled the conjecture related to the existence of symplectic structures. Donaldson theory naturally leads to the instanton Floer homology of a 3-manifold. Similarly, the Seiberg-Witten theory leads to the Floer homology of Chern-Simons functional defined by a spin^c manifold (see [2]). It is wellknown that the Seiberg-Witten equation has no reducible solutions if the line bundle L has nontrivial first Chern class. M. Marcolli defined the Seiberg-Witten-Floer homology of 3-manifold in these cases. B. L. Wang gave the definition of the Seiberg-Witten-Floer (SWF) homology of homology 3-sphere where any line bundle has trivial first Chern class. In this paper, we will apply the Morse-Bott theory to the Chern-Simons functional and define the $U(1)$ -equivariant SWF homology and cohomology of Z -homology 3-sphere.

1 Preliminaries

Let $f: M \to \mathbf{R}$ be an invariant smooth function on a Riemannian G -manifold M. In this section, we will give some results about Morse-Bott theory. For full details we refer to [3]. Let α be a critical orbit of f. Define the rank of the negative normal bundle as $i(\alpha)$. Denote the critical orbits by $\alpha, \beta, \cdots$. Due to the presence of the Riemannian metric we can consider the gradient flow of f. Let $\mathscr{M}_0(\alpha, \beta)$ be the moduli space of the gradient lines tending to α for $t \to -\infty$ and to β for $t \to \infty$. The gradient lines are parametrized in the sense that if $\gamma(t) \in \mathscr{M}_0(\alpha, \beta)$ then for $t_0 \neq 0$ the line $\gamma(t + t_0)$ represents a different element of $\mathscr{M}_0(\alpha, \beta)$. The quotient by the $\mathbf{R}$- action is the reduced moduli space denoted by $\widetilde{\mathscr{M}}_0(\alpha, \beta)$. When the moduli spaces are cut out transversely, that is, when they are transversal intersections of the stable manifold of β and the unstable manifold of α, it follows that:

$$\dim \widetilde{\mathscr{M}}_0(\alpha, \beta) = \dim\alpha + i(\alpha) - i(\beta) - 1$$

The moduli spaces come equipped with smooth G -equivariant upper and lower end-point maps

$$e_\alpha^+ : \widetilde{\mathscr{M}}_0(\alpha, \beta) \to \alpha, e_\beta^- : \widetilde{\mathscr{M}}_0(\alpha, \beta) \to \beta$$

obtained by following a gradient line as it converges to a point on a critical orbit. We can compactify the moduli spaces by adding strata of factorisations of gradient lines through intermediate critical points (compare [3])

$$\overline{\widetilde{\mathscr{M}}_0}(\alpha, \beta) = \bigcup \widetilde{\mathscr{M}}_0(\alpha, \gamma_1) \times_{\gamma_1} \widetilde{\mathscr{M}}_0(\gamma_1, \gamma_2) \times_{\gamma_2} \cdots \times_{\gamma_{n-1}} \widetilde{\mathscr{M}}_0(\gamma_{n-1}, \beta)$$

where the union is over (possibly empty) sequences of critical orbits γ_i with $i(\beta) < i(\gamma_{n-1}) < \cdots < i(\alpha)$. The fibred products in this compactfication ensure that a broken gradient line will rest and resume from the same point on an intermediate critical orbit. The compactified moduli space has the structure of a manifold with corners where the number of breaking points gives the codimension of the stratum. Moreover, the endpoint maps extend smoothly over the lower strata,

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and coincide with e_α^+ on $\widetilde{\mathcal{M}}_0(\alpha,\gamma_1)$ and e_β^- on $\widetilde{\mathcal{M}}_0(\gamma_{n-1},\beta)$. Let f be an invariant function with transverse moduli spaces. The critical orbits and moduli spaces may be oriented such that the orientation of the codimension one stratum of the compactification above agrees with its orientation as a boundary. We define a co-complex

$$C_G^k = \bigoplus_{i(\alpha)+i=k}\Omega_G^i(\alpha)$$

with differential given by a matrix

$$(d_{\mathscr{G}})_{\alpha,\beta}\omega = \begin{cases} d_G\omega & \text{if } \alpha=\beta \\ (-1)^{r(\omega)}(e_\alpha^+)_*\circ (e_\beta^-)^*\omega & \text{if } i(\alpha)>i(\beta) \\ 0 & \text{otherwise}\end{cases}$$

where the map $(e_\alpha^+)_*$ is the integration along the fiber of the bundle $e_\alpha^+:\widetilde{\mathcal{M}}_0(\alpha,\ \beta)\to\alpha$ and the complex

$$C_{G,k} = \bigoplus_{I(\alpha)+i=k}\Omega_{G,i}(\alpha)$$

with differential

$$(\partial_{\mathscr{G}})_{\alpha,\beta}\eta = \begin{cases} \partial_G\eta & \text{if } \alpha=\beta \\ (-1)^{r(\eta)}(e_\beta^-)_*\circ (e_\alpha^+)^*\eta & \text{if } i(\alpha)>i(\beta) \\ 0 & \text{otherwise}\end{cases}$$

where $r(\omega)$ and $r(\eta)$ are integers known as the form ranks of ω and η and d_G and ∂_G are the differentials of the ordinary equivariant cohomology and homology respectively. For an equivariant form v , we define the form rank $r(v)$ to be the maximal degree of differential form appearing in the expression of v . It is easy to see that we have the following version of Stokes theorem.

Lemma 1　Let $\pi:E\to B$ be a fibre bundle with fibre F , a compact manifold with boundary. Denote by π_∂ the fibre bundle restricted to the boundary of F . Then for $\omega\in\Omega^k(E)$,

$$\pi_*(d\omega) = d\pi_*(\omega) + (-1)^l(\pi_\partial)_*(\omega)$$

where $l=r(\omega)-d+1$ and $d=\dim F$. The pairing between equivariant cohomology and homology can be defined as follows (see [3]):

$$\langle \oplus_\alpha X_\alpha\otimes\eta_\alpha,\ \oplus_\beta X_\beta\otimes\zeta_\beta\oplus\rangle_M = \sum_\alpha\langle X_\alpha\otimes\eta_\alpha,\phi_\alpha\otimes\zeta_\alpha\rangle_\alpha$$

Lemma 2[4]　For $G=U(1)$, $(d_{\mathscr{G}})_{\alpha,\beta}=0$, $(\partial_{\mathscr{G}})_{\alpha,\beta}=0$, $(d_{\mathscr{G}})^2_{\alpha,\beta}=0$, $(\partial_{\mathscr{G}})^2_{\alpha,\beta}=0$, if $i(\alpha)-i(\beta)>2$. Let Y be a compact, connected, closed, oriented Z - homology 3 -sphere. Every spin^c structure on Y induces a spin^c structure on the 4 -manifold $X=Y\times\mathbf{R}$ by pullback; that is, there are two spin^c bundles S^+ and S^- and they can be identified *via* the Clifford multiplication by $\mathrm{d}t$, both can be identified with the spin^c bundle S on Y . Since Y is a spin manifold, we can write the determinant bundle of S as L^2 for some line bundle L on Y . The Seiberg-Witten equations for $Y\times\mathbf{R}$ are the equations for a unitary $U(1)$ -connection A on L and a spinor section $\psi\in\Gamma(S)$:

$$F_A^+ = 1/4\langle e_ie_j\psi,\psi\rangle e^i\wedge e^j,\ D_A\psi=0 \quad (1)$$

where F_A^+ is the self-dual part of the curvature of A , and $e_i\mid_{i=1}^4$ is the orthonormal frame for $T(Y\times\mathbf{R})$ acting on spinors by Clifford multiplication. Write $e^i\mid_{i=1}^4$ as the dual basis elements for $T^*(Y\times\mathbf{R})$. Denote the Hermite inner product by $\langle,\rangle$ and sometimes write $\bar\psi_1\psi_2=\langle\psi_1,\psi_2\rangle$. Here D_A is the Dirac operator on $T(Y\times\mathbf{R})$ associated with the connection A . We say that a pair (A,ψ) is in temporal gauge if the $\mathrm{d}t$ -component of A vanishes identically. In this paper, we assume that (A,ψ) is in temporal gauge. So we only consider the gauge group $\mathscr{G}=\mathrm{Map}(Y,u(1))$. It is easy to see that equation (1) is invariant under the gauge group $\mathscr{G}$. Denote the space of all (A,ψ) by $\mathscr{A}$ and $\mathscr{B}=\mathscr{A}/\mathscr{G}$.

Definition 1　For a fixed unitary connection B on L , define the functional $CS:\mathscr{A}\to\mathbf{R}/\mathbf{Z}$ as follows:

$$CS(A,\psi) = \int_Y(A-B)\wedge F_A + \int_Y\langle\psi,\partial_A\psi\rangle\mathrm{dvol}(Y)$$

where $\langle,\rangle$ denots the ordinary L^2 inner product.

For Z -homology sphere, it is easy to prove that CS is a $\mathscr{G}$-invariant functional. So CS can be reduced to the quotient space $\mathscr{B}$. Fix a base point $y_0\in Y$ and define the group of based gauge transformations as $\mathscr{G}_0=\{g\in\mathscr{G}\mid g(y_0)=\mathrm{id}\}$. Then $\mathscr{G}_0$ acts freely on $\mathscr{A}$ and we denote the orbit space by $\mathscr{B}_0$. On $\mathscr{B}_0$ there remains an action of $\mathscr{G}/\mathscr{G}_0\cong U(1)$. In this paper, we will use the subgroup $\widetilde{\mathscr{G}}_0$ of based gauge transformations of degree 0. Therefore, on the quotient space $\widetilde{\mathscr{B}}_0$, we have a real-valued $\mathbf{Z}\times U(1)$ -equivariant function CS. The $\mathbf{Z}$-action is free and we shall not mention it unless necessary. We will primarilly study CS as a $U(1)$ -invariant function on $\widetilde{\mathscr{B}}_0$. Moreover, we need to use the usual Sobolev completion of the spaces of connections and gauge transformations to make sure that $\widetilde{\mathscr{B}}_0$ is a Banach manifold. We say that the solution $(A,\psi)\in\mathscr{A}$ of (1) is irreducible if $\psi\neq0$. Denote by $\mathscr{B}^*$ the space of irreducible pairs (A,ψ) modulo the gauge group $\mathscr{G}$. Denote by $\widetilde{\mathscr{B}}_0^*$ the quotient space of the irreducible pairs (A,ψ) modulo the action of the gauge group $\widetilde{\mathscr{G}}_0$. Then $\widetilde{\mathscr{B}}_0^*$ is a smooth Hilbert manifold. Now, we introduce the perturbed Chern-Simons functional CS_α as follows:

$$CS_\alpha(A,\psi) = \int_Y (A-B)\wedge(F_A - 2\mathrm{d}\alpha) + \int_Y \langle \psi, \partial_A \psi\rangle \mathrm{dvol}(Y) \quad (3)$$

where $\alpha \in \Omega^1(Y, \mathrm{i}\mathbf{R})$. Furthermore, its critical point is the solution of the perturbed Seiberg-Witten equation on Y:

$$* F_A = q(\psi) + * \mathrm{d}\alpha,\quad \partial_A\psi = 0 \quad (4)$$

where $q(\psi)$ is the quadratic function of ψ, in local coordinates $q(\psi) = 1/2\langle e_i \cdot \psi, \psi\rangle e^i$.

Now the perturbed Seiberg-Witten equation on $Y \times \mathbf{R}$ (that is, the gradient flow equation) is

$$\frac{\partial A}{\partial t} = * F_A - q(\psi) - * \mathrm{d}\alpha, \frac{\partial \psi}{\partial t} = \partial_A \psi \quad (5)$$

Define the moduli space $\mathscr{M}_{Y,\alpha}$ to be the solution space of (4) modulo the action of $\mathscr{G}$, and denote the moduli space as $\mathscr{M}^*_{Y,\alpha}$ for the irreducible solution space of (4) modulo the action of $\mathscr{G}$. We have

Proposition 1 For any metric g on Y, there is an open dense set in $* \mathrm{d}(\Omega^1_{L^2_2}(Y, \mathrm{i}\mathbf{R})$, such that $\mathscr{M}^*_{Y,\alpha}$ is a smooth 0-dimensional manifold and the only reducible solution $(\alpha, 0)$ is isolated for any $* \mathrm{d}\alpha$ in this set.

We denote the moduli space of finite energy solutions for (5) which connect the two critical points a, b as

$$\mathscr{M}(a,b) = \left\{\gamma(t)\left|\begin{array}{l}① \ \gamma(t) \text{ is the finite energy solution of} \\ \quad (5) \text{ modulo the gauge group } \mathscr{G} \\ ② \ \lim_{t\to-\infty}\gamma(t) = a \\ ③ \ \lim_{t\to+\infty}\gamma(t) = b\end{array}\right.\right\}$$

Since there is a natural $\mathbf{R}$-action on $\mathscr{M}(a,b)$ (time translation), we define the $\mathbf{R}$-quotient space by $\mathscr{M}(a,b) = \mathscr{M}(a,b)/\mathbf{R}$. Similar to [4], we may assign an index $i(a)$ to a critial orbit a of the perturbed Chern-Simons functional and prove the following results:

Proposition 2 Let $a, b, c \in M_{Y,\alpha}$ with $i(a) > i(b) > i(c)$. Suppose b is irreducible. Then for T sufficiently large, there is an embedding map g: $\mathscr{M}(a,b) \times \mathscr{M}(b,c) \times [T, \infty) \to \mathscr{M}(a,c)$. If b is the reducible critical point, then there is a gluing group $U(1)$ arising from the stablisher of b in the gluing map; this means that there is a local diffeomorphism,

$$\hat{g}: \mathscr{M}(a,b) \times \mathscr{M}(b,c) \times U(1) \times [T, \infty) \to \mathscr{M}(a,c).$$

Proposition 3 Suppose a, c are two irreducible critical points with index given by $i(a) = i(c) + 2$. Then the boundary of $\mathscr{M}(a,c)$ consists of the union $\bigcup_{\{b \mid i(b) = i(a)-1\}} \mathscr{M}(a,b) \times_b \mathscr{M}(b,c)$ where b runs over the set of the irreducible critical point.

2 Equivariant Seiberg-Witten-Floer homology

Our purpose of this section is to define the $U(1)$-equivariant SWF homology, their pairing and module structure for homology 3-sphere. Let Y be a compact, connected, closed, oriented $\mathbf{Z}$-homology 3-sphere and assume that we have chosen a generic perturbation functional CS_α to CS defined in previous section. From the previous section, we know that the module spaces of gradient lines $\mathscr{M}_0(a,b)$ between the critical orbits a, b of CS_α are smooth $U(1)$-manifolds. These can be studied as framed monopole moduli spaces on the cylinder $Y \times \mathbf{R}$, modulo the translation action. In fact, we can assign an index $i(a)$ to each critical orbit such that $\dim \mathscr{M}_0(a,b) = i(a) - i(b) + \dim a - 1$. Moreover, we have equivariant endpoint maps

$$e_a^+ : \mathscr{M}_0(a,b) \to a,\ e_b^+ : \mathscr{M}_0(a,b) \to b$$

Gradient lines only flow from critical orbits with higher indices to ones with lower indices. Let g be the Lie algebra of $G = U(1)$ and denote by $S^*(g)$ the symmetric algebra on g, where elements in g have degree 2. Now we introduce the $S^*(g)^G$-modules $CF_{\mathscr{G};k}(Y)$ and $CF^k_{\mathscr{G}}(Y)$ together with the differential operators $\delta_{\mathscr{G}}$ and $D_{\mathscr{G}}$ respectively as follows:

$$CF_{\mathscr{G};k}(Y) = \bigoplus_{i(a)+j=k}\Omega_{G,j}(a)$$

$$CF^k_{\mathscr{G}}(Y) = \bigoplus_{i(a)+j=k}\Omega^k_j(a)$$

$$(\delta_{\mathscr{G}})_{a,b}\eta = \begin{cases}\partial_G \eta & \text{if } a = b \\ (-1)^{r(\eta)}(e_b^-)_* \cdot (e_a^+)^* \eta & \text{if } i(a) > i(b) \\ 0 & \text{otherwise}\end{cases}$$

$$(D_{\mathscr{G}})_{a,b}\omega = \begin{cases}d_G \omega & \text{if } a = b \\ (-1)^{r(\omega)}(e_a^+)_* \cdot (e_b^-)^* \omega & \text{if } i(a) = i(b) \\ 0 & \text{otherwise}\end{cases}$$

where the integer $r(\eta)$ equals to the form rank of η (see Section 1); $\Omega_{G,*}(a)$ and $\Omega^k_G(a)$ are defined as in Section 1.

Proposition 4 $\delta^2_{\mathscr{G}} = 0$ and $D^2_{\mathscr{G}} = 0$.

Proof We will deal with the case of cohomology; the case of homology is formally the same. Let $\omega \in \Omega^*_G(a)$, where a is the critical orbit of CS_α. We need to prove that for every critical orbit c the component $(D^2_{\mathscr{G}}\omega)_c$ vanishes. If $c = a$, then the conclusion follows from $d_G^2 = 0$. Now assume $c \neq a$. We need to show that $\sum_b (D_{\mathscr{G}})_{c,b}(D_{\mathscr{G}})_{b,a} = 0$. Ignoring signs in the differential, this amounts to

$$\sum_{i(a)<i(b)<i(c)} (e_c^+)_*(e_b^-)^*(e_b^+)_*(e_a^-)^*\omega +$$
$$(e_c^+)_*(e_a^-)^* d_G\omega + d_G(e_b^+)_*(e_a^-)^*\omega = 0 \quad (6)$$

To prove (6), we first consider the structure of the moduli space $\widetilde{\mathscr{M}}_0(a,b)$. From Proposition 2 and 3, it follows that

① $\mathscr{M}_0(a,b) \to a$ is an equivariant smooth fibration of dimension dim $a + i(a) - i(b)$ for all moduli spaces to which we apply a push forward.

② The codimension 1 boundary of $\mathscr{M}_0(a,c)$ equals to $\bigcup_b \mathscr{M}_0(a,b) \times_b \mathscr{M}_0(b,c)$.

If we consider the form $(e_a^-)^*\omega$ on the fibration $\mathscr{M}_0(a,c) \to a$ whose fibre is a manifold with boundary, then (6) follows from Lemma 1.

Now, we may give the definition of equivariant SWF homology as follows:

Definition 2　Let Y be a compact, connected, closed, oriented, **Z**-homology 3-sphere. Define the equivariant SWF homology and cohomology of Y to be the $S^*(g^*)^G$ modules $HF^{SW}_{\mathscr{G},*}(Y) \equiv H_*(CF_{\mathscr{G},*}, \delta_{\mathscr{G}})$ and $HF^{*\,SW}_{\mathscr{G}}(Y) \equiv H^*(CF^*_{\mathscr{G}}, D_{\mathscr{G}})$, respectively. Similar to the case of equivariant Floer homology, we can get the pairing on equivariant SWF groups as follows.

Proposition 5　For Y , we have $\langle \delta_{\mathscr{G}}\eta, \omega\rangle = \langle \eta, D_{\mathscr{G}}\omega\rangle$ which means that the pairing descends to the equivariant SWF homology and cohomology. Moreover, the action of $S^*(g^*)^G$ is symmetric with respect to this pairing. The proof of this proposition is similar to that of proposition 4.3 of [3].

Next, we note that the equivariant homology and cohomology are $S^*(g^*)^G$-modules. Therefore, we define the module structure as a chain map on the SWF complexes through the direct sum, i.e.

$$CF_{\mathscr{G},*} = \bigoplus_a \Omega_{G,*}(a), CF^*_{\mathscr{G}} = \bigoplus_a \Omega^*_G(a)$$

where multiplication by an invariant polynomial $\phi \in S^*(g^*)^G$ is just given by multiplication in each component. Since the higher order terms of the boundary operators are defined purely in terms of the differential form part of an equivariant differential form, one sees that multiplication by an invariant polynomial commutes with the boundary operator. Therefore, this gives the module structures on equivariant Seiberg-Witten-Foler homology and cohomology.

3　Equivariant SWF cohomology and homology are topological invariants

In this section, we will use Floer's idea to relate the definitions for different choices and prove that the equivariant SWF homology and cohomology are independent of metrics, perturbation one-form and cobordisms. Suppose that there are two metrics g_0, g_1 on Y with generic perturbing forms α_0, α_1 respectively such that their SWF cohomology $HF^{*\,SW}_{\mathscr{G}}(Y,g_0,\alpha_0)$, $HF^{*\,SW}_{\mathscr{G}}(Y,g_1,\alpha_1)$ from the Floer complexes $(CF^*_{\mathscr{G}}(Y,g_0,\alpha_0), D_{\mathscr{G}})$, $(CF^*_{\mathscr{G}}(Y,g_1,\alpha_1), D_{\mathscr{G}})$ are well-defined. The coresponding critical sets and various moduli spaces are written as $M^*_{Y,\alpha_0}, M^*_{Y,\alpha_1}$, $\mathscr{M}(a_0,b_0), \mathscr{M}(a_1,b_1)$, respectively, and so on. Choose a path $g_t, 0 \leqslant t \leqslant 1$, of metrics on Y connecting g_0, g_1 . We consider a complete 4-manifold $W = Y \times \mathbf{R}$ with metric being a product metric outside [0, 1] and $g_t + \mathrm{d}t^2$ on $Y \times [0,1]$. Then the perturbed Seiberg-Witten-Floer equation in temporal gauge on $W = Y \times \mathbf{R}$ is

$$\frac{\partial A}{\partial t} = *_t F_A - q(\psi) - *_t \mathrm{d}\alpha_t, \quad \frac{\partial \psi}{\partial t} = \partial_A \psi$$

where $*_t$ is the complex linear Hodge star on $(Y \times \{t\}, g_t)$ and α_t is a generic path of the perturbing one-form connecting α_0, α_1 . We will denote the moduli space of solutions connecting a critical point a_0 of CS_{α_0} and a_1 of CS_{α_1} by $\mathscr{M}_F(a_0,a_1)$. The endpoint maps on the spaces $\mathscr{M}_F(a_0,a_1)$ defined by all critical points a_0, a_1 define a matrix of operators

$$C_{a_0,a_1}: \Omega^*_G(a_1) \to \Omega^*_G(a_0)$$
$$\omega \mapsto (e^+_{a_0})_*(e^-_{a_1})^*\omega$$

what we use to define a map $C: CF^k_{\mathscr{G}}(Y,g_1,\alpha_1) \to CF^k_{\mathscr{G}}(Y,g_0)$. Similarly, we obtain a map on the equivariant SWF homology complex. We will now prove the following proposition:

Proposition 6　The map C is well-defined, commutes with the $S^*(g^*)^G$ action and we have the following conclusions:

① C is a co-chain map: $D_{\mathscr{G}}C + CD_{\mathscr{G}} = 0$;

② If the perturbation and metric are constant (i.e. $g_t = g_0, \alpha_t = \alpha_0$) then $C = $ id on cohomology.

To prove this proposition, we first need to describe the structure of the moduli space $\mathscr{M}_F(a_0,a_1)$.

Proposition 7　For a generic path α_t , $\mathscr{M}_F(a_0, a_1)$ is a compact smooth manifold with the dimension given $i(a_0) - i(a_1)$. Moreover, given the orientation on $W = Y \times \mathbf{R}$, there is a natural orientation on $\mathscr{M}_F(a_0, a_1)$ such that the following gluing maps are orientation preserving local diffeomorphisms:

$$g: \mathscr{M}_0(a_0, a_0') \times (-\infty, -T] \times \mathscr{M}_F(a_0', a_1) \to \mathscr{M}_F(a_0, a_1)$$

$$g: \mathscr{M}_F(a_0, a_1') \times [T, \infty) \times \mathscr{M}_0(a_1', a_1) \to \mathscr{M}_F(a_0, a_1)$$

onto ends of $\mathscr{M}_F(a_0, a_1)$, where T is a sufficiently large positive number.

The proof of this proposition is a standard gluing argument. Therefore, we omit it. The following corollary immediately follows from the above proposition

Corollary 1 Suppose $a_0 \in M^*_{Y,\alpha_0}$ and $a_1 \in M^*_{Y,\alpha_1}$ with index $i(a_0) = i(a_1)$. Then the oriented codimension one boundary of $\mathscr{M}_F(a_0, a_1)$ is

$$\bigcup_{a_0'} \mathscr{M}_0(a_0, a'_0) \times_{a_0'} \mathscr{M}_F(a_0', a_1) \cup \bigcup_{a_1'} \mathscr{M}_F(a_0, a_1') \times_{a_1'} \mathscr{M}_0(a_1', a_1)$$

where a_0', a_1' run over the set of the irreducible critical sets M^*_{Y,α_0}, M^*_{Y,α_1} respectively.

Proof of proposition 6 From lemma 2, we know that the matrix elements of operator C are zero if the dimension, $i(a_0) - i(a_1)$, of $\mathscr{M}_F(a_0, a_1)$ is large. Therefore, we only need to consider the moduli spaces with the compactification of corollary 4.3. Thus the operator C is well-defined. From the definition of C, we know that C commutes with $S^*(g^*)^G$-action.

(1) Let $\omega \in \Omega^*_C(a_1)$, where a_1 is a critical orbit of CS_{α_1}. We need to show that for every critical point a_0 we have $(D_{\mathscr{G}} C\omega)_{a_0} + (CD_{\mathscr{G}}\omega)_{a_0} = 0$. Ignoring signs in the defferential, this amounts to

$$\sum_{a_0'} (e^+_{a_0})_* (e^-_{a_0'})^* (e^+_{a_0'})_* (e^-_{a_1})^* \omega + \sum_{a_1'} (e^+_{a_0})_* (e^-_{a_1'})^* (e^+_{a_1'})_* (e^-_{a_1})^* \omega +$$

$$d_G (e^+_{a_0})_* (e^-_{a_1})^* \omega + (e^+_{a_0})_* (e^-_{a_1'})^* d_G \omega = 0 \quad (7)$$

Consider the form $(e^-_{a_1})^* \omega$ on the fibrations $\mathscr{M}_F(a_0', a_1) \to a_0'$ and $\mathscr{M}_0(a_1', a_1) \to a_1'$ whose fibres are manifolds with boundaries. Proposition 7 guarantees that we may apply Stokes' theorem to our cases. Therefore, (7) follows directly from Stokes' theorem for a fibration with boundary, Lemma 1. This proves that C is a co-chain map.

(2) If the perturbation and metric are independent of time, then there is a translation action on the moduli space. The vector field of this action is a fibre for both e_{a_0} and e_{a_1} and, as a result, the only non-trivial contributions come from translation invariant monopoles. These are constant and this shows that $C = \mathrm{id}$.

To prove that the $U(1)$-equivariant SWF homology and cohomology are topological invariants, we need to prove the following proposition.

Proposition 8 Orientation preserving deffeomorphisms act nataurally on the equivariant SWF groups.

Proof Let h be an orientation-preserving diffeomorphism of Y. Then the map h induces a diffeomorphism $\tilde{h}: \mathscr{B}_0 \to \mathscr{B}_0$. Suppose that α is an appropriate generic perturbation one-form of the Chern-Siomons functional. Then $\tilde{h}(\alpha)$ gives another generic perturbation one-form of the Chern-Simons functional. Moreover, $\tilde{h}$ maps critical orbits and gradient flow lines of CS_α to those of $CS_{\tilde{h}(\alpha)}$. Therefore, the diffeomorphism h induces a map between the two complexes defined by the two perturbation one-forms. From Proposition 6, it follows that the homology of these complexes are isomorphic and thus we obtain a natural map on the equivariant SWF groups.

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同调三维球面的 $U(1)$ 等变 Seiberg-Witten-Floer 同调群

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摘　要: 通过对 Chern-Simons 泛函应用 Morse-Bott 理论, Z-同调三维球面的等变 Seiberg-Witten-Floer 下同调群和上同调群, 以及它们的配对和模结构能够被定义, 并且还可证明它们是拓扑不变量.

关键词: $U(1)$ 等变; Seiberg-Witten-Floer 同调; 拓扑不变量