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# 齐型空间上的 Besov 空间的一种特征刻划*

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摘 要: 通过恒等逼近及 Calderon 再生公式, 给出了齐型空间上的 Besov 空间 $B_p^{\alpha p}$ 在 $0<\alpha<1$, $1<p<\infty$ 的一种新的刻划.

关键词: 齐型空间; 恒等逼近; Calderon 型再生公式; Besov 空间
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## 1 基本概念

齐型空间最早是由 Coifman 与 Weiss 引入的[1].

称 $\delta$ 是 $X$ 上的一个拟度量, 如果 $\delta: X\times X\to[0,\infty)$ 满足下面条件

(1) $\delta(x,y)=0$ 当且仅当 $x=y$.

(2) $\delta(x,y)=\delta(y,x)$, $\forall x,y\in X$.

(3) $\exists$ 常数 $A<\infty$, 使对 $\forall x,y,z\in X$, 有

$$\delta(x,z)\leqslant[\delta(x,y)+\delta(y,z)]$$

定义 1 称 $(X,\delta,\mu)$ 为一个齐型空间, 如果 $\delta$ 是 $X$ 上的一个拟度量, $\mu$ 是非负测度, 且 $\forall x\in X$, $r>0$, $0<\mu(B(x,r))<\infty$, 同时满足

$$\mu(B(x,2r))\leqslant A'[\mu(B(x,r))]$$

其中, $A'$ 是常数.

齐型空间包括了 Lipschitz 曲线与曲面, 及 Riemann 流形等许多常见空间. 由于 $X$ 上没有平移与展缩的群结构, 所以 $X$ 上的调和分析需要有不同于 $\mathbf{R}^n$ 的方法与技巧.

Macias 和 Segovia 证明了[2], 可以用另一个拟度量 $\rho$ 代替 $\delta$, 使空间 $X$ 的拓扑不变, 其中 $\rho$ 满足: $\exists$ 常数 $C>0$ 与 $0<\theta<1$, 对 $\forall x,x',y\in X$, 有

$$\rho(x,y)\sim\inf\{\mu(B): x,y\in B\}$$

$$|\rho(x,y)-\rho(x',y)|\leqslant C\rho(x,x')^{\theta}[\rho(x,y)+\rho(x',y)]^{1-\theta}$$

齐型空间上的 Besov 空间, 首先是由韩永生[3,4]引进的.

定义 2 算子列 $\{S_k\}_{k\in\mathbf{Z}}$ 称为恒等逼近, 假如存在 $0<\delta,\varepsilon\leqslant\theta$ 及 $C<\infty$, 使对所有 $x,x',y,y'\in X$, $S_k$ 的核 $S_k(x,y): X\times X\to C$ 满足

$$|S_k(x,y)|\leqslant C\frac{2^{-k\varepsilon}}{(2^{-k}+\rho(x,y))^{1+\varepsilon}} \tag{1}$$

$$|S_k(x,y)-S_k(x',y)|\leqslant C\left(\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}\right)^{\delta}\frac{2^{-k\varepsilon}}{(2^{-k}+\rho(x,y))^{1+\varepsilon}}$$

$$\forall\rho(x,x')\leqslant\frac{1}{2A}(2^{-k}+\rho(x,y)) \tag{2}$$

$$|S_k(x,y)-S_k(x,y')|\leqslant C\left(\frac{\rho(y,y')}{2^{-k}+\rho(x,y)}\right)^{\delta}\frac{2^{-k\varepsilon}}{(2^{-k}+\rho(x,y))^{1+\varepsilon}}$$

$$\forall\rho(y,y')\leqslant\frac{1}{2A}(2^{-k}+\rho(x,y)) \tag{3}$$

$$\int S_k(x,y)\,\mathrm{d}\mu(y)=1\quad\forall x\in X \tag{4}$$

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$$\int S_k(x,y)\,\mathrm{d}\mu(x)=1\quad \forall y\in X \tag{5}$$

定义3 一个恒等逼近$\{S_k\}_{k\in\mathbf{Z}}$称为满足条件(H),假如

$$|S_k(x,y)-S_k(x',y)-S_k(x,y')+S_k(x',y')|\leqslant$$
$$C\left(\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}\right)^{\delta}\left(\frac{\rho(y,y')}{2^{-k}+\rho(x,y)}\right)^{\delta}\frac{2^{-k\varepsilon}}{(2^{-k}+\rho(x,y))^{1+\varepsilon}}$$
$$\forall\rho(x,x')\leqslant\frac{1}{2A}(2^{-k}+\rho(x,y)),\qquad \forall\rho(y,y')\leqslant\frac{1}{2A}(2^{-k}+\rho(x,y)) \tag{6}$$

引理1 假设$\{S_k\}_{k\in\mathbf{Z}}$是一个恒等逼近,令$D_k=S_k-S_{k-1}$对$\forall k\in Z$,则对$\delta'\leqslant\delta$,$\delta'\leqslant\varepsilon$,存在仅依赖于$\delta$,$\delta'$和$\varepsilon$,但不依赖于$k$和$l$的常数$C$,使得

$$|D_kD_l(x,y)|\leqslant C2^{-|l-k|\delta'}\frac{2^{-(k\wedge l)\varepsilon}}{(2^{-(k\wedge l)\varepsilon}+\rho(x,y))^{1+\varepsilon}} \tag{7}$$

下面的引理是齐型空间上的Calderon型再生公式[3~6].

引理2 假设$\{S_k\}_{k\in\mathbf{Z}}$是一恒等逼近且满足条件(H),在定义2,3中$\delta=\varepsilon\leqslant\theta$,令$D_k=S_k-S_{k-1}$,则存在算子列$\{D_k\}_{k\in\mathbf{Z}}$,及$\{\overline{D_k}\}_{k\in\mathbf{Z}}$,使对$\forall f\in(m_0(\beta,r))'$[3],有

$$f=\sum_k D_kD_k(f)=\sum_k\overline{D_k}D_k(f) \tag{8}$$

这里级数收敛的意义参见文[3].

现在可以叙述齐型空间上的Besov空间的定义了.

定义4 设$-\varepsilon<\alpha<\varepsilon$,$\{S_k\}_{k\in\mathbf{Z}}$是一恒等逼近,且满足条件$(H)$,令$D_k=S_k-S_{k-1}$,定义齐型空间上的Besov空间$B_q^{\alpha,p}$为所有满足下列不等式的分布$f\in(m_0(\beta,r))'$组成的空间

$$\|f\|_{B_q^{\alpha,p}}=\{\sum_k(2^{k\alpha}\|D_k(f)\|_p)^q\}^{1/q}<\infty \tag{9}$$

其中,$\max\left(\frac{1}{1+\varepsilon},\ \frac{1}{1+\varepsilon+\alpha}\right)<p\leqslant\infty$,$0<q\leqslant\infty$.

由Calderon型再生公式与Plancherel-Polya不等式可知,Besov空间与$\{S_k\}$的选取无关[3,5,6].

## 2 结论及其证明

本文的目的是给出齐型空间上的Besov空间$B_p^{\alpha,p}$在$0<\alpha<1$,$1<p<\infty$的一个等价特性,即证明下面的定理.

定理1 假设$0<\alpha<1$,$1<p<\infty$,则对$\forall f\in L^p$,

$$\|f\|_{B_p^{\alpha,p}}\sim\left\{\iint\frac{|f(z)-f(\omega)|^p}{\rho(z,\omega)^{1+\alpha p}}|\mathrm{d}\mu(z)||\mathrm{d}\mu(\omega)|\right\}^{1/p} \tag{10}$$

证明:设$f\in L^p$

$$\iint\frac{|f(z)-f(\omega)|^p}{\rho(z,\omega)^{1+\alpha p}}|\mathrm{d}\mu(z)||\mathrm{d}\mu(\omega)|<\infty$$

则 $\|D_k(f)\|_p^p=\|\int D_k(z,\omega)f(\omega)\mathrm{d}\mu(\omega)\|_p^p=\|\int D_k(z,\omega)(f(\omega)-f(z))\mathrm{d}\mu(\omega)\|_p^p=$

$$\int\left|\int D_k(z,\omega)(f(\omega)-f(z))\mathrm{d}\mu(\omega)\right|^p|\mathrm{d}\mu(z)|\leqslant C\iint|D_k(z,\omega)||f(\omega)-f(z)|^p|\mathrm{d}\mu(\omega)||\mathrm{d}\mu(z)|$$

因此
$$\|f\|_{B_p^{\alpha,p}}=\sum_{k\in\mathbf{Z}}(2^{k\alpha}\|D_k(f)\|_p)^p\leqslant$$
$$\sum_{k\in\mathbf{Z}}C\iint 2^{k\alpha p}|D_k(z,\omega)||f(\omega)-f(z)|^p|\mathrm{d}\mu(\omega)||\mathrm{d}\mu(z)|\leqslant$$
$$C\iint\frac{|f(\omega)-f(z)|^p}{\rho(z,\omega)^{1+\alpha p}}|\mathrm{d}\mu(z)||\mathrm{d}\mu(\omega)| \tag{11}$$

这是由于我们可选择$\varepsilon>\alpha p$,使得

$$\sum_{k\in\mathbf{Z}} 2^{k\alpha p} |D_k(z,\omega)| \leqslant \sum_{k\in\mathbf{Z}} 2^{k\alpha p} \frac{2^{-k\varepsilon}}{(2^{-k}+\rho(z,\omega))^{1+\varepsilon}} \leqslant \sum_{k\in\mathbf{Z}} 2^{-k(\varepsilon-\alpha p)} \frac{1}{\rho(z,\omega)^{1+\alpha p}} \leqslant \frac{C}{\rho(z,\omega)^{1+\alpha p}}$$

反过来，容易验证 $\sum D_k = I$，即 $f(z) = \sum_{k\in\mathbf{Z}} D_k(f)(z)$。因此

$$f(z)-f(\omega) = \sum_{k\in\mathbf{Z}} [D_k(f)(z)-D_k(f)(\omega)]$$

故

$$\iint \frac{|f(z)-f(\omega)|^p}{\rho(z,\omega)^{1+\alpha p}} |\mathrm{d}\mu(z)||\mathrm{d}\mu(\omega)| \leqslant$$

$$\int\sum_{k\in\mathbf{Z}}\int_{2^{-k}\leqslant\rho(z,\omega)<2^{-k+1}} \frac{|f(z)-f(\omega)|^p}{\rho(z,\omega)^{1+\alpha p}} |\mathrm{d}\mu(z)||\mathrm{d}\mu(\omega)| \leqslant$$

$$C\int\sum_{k\in\mathbf{Z}}\int_{2^{-k}\leqslant\rho(z,\omega)<2^{-k+1}} \frac{\left|\sum_{j\geqslant k} D_j(f)(z)\right|^p}{\rho(z,\omega)^{1+\alpha p}} |\mathrm{d}\mu(z)||\mathrm{d}\mu(\omega)| +$$

$$C\int\sum_{k\in\mathbf{Z}}\int_{2^{-k}\leqslant\rho(z,\omega)<2^{-k+1}} \frac{\left|\sum_{j\geqslant k} D_j(f)(\omega)\right|^p}{\rho(z,\omega)^{1+\alpha p}} |\mathrm{d}\mu(z)||\mathrm{d}\mu(\omega)| +$$

$$C\int\sum_{k\in\mathbf{Z}}\int_{2^{-k}\leqslant\rho(z,\omega)<2^{-k+1}} \frac{\left|\sum_{j<k} [D_j(f)(z)-D_j(f)(\omega)]\right|^p}{\rho(z,\omega)^{1+\alpha p}} |\mathrm{d}\mu(z)||\mathrm{d}\mu(\omega)| = \mathrm{I}+\mathrm{II}+\mathrm{III}$$

$$\mathrm{I} \leqslant C\int\sum_{k\in\mathbf{Z}} 2^{k\alpha p} \left|\sum_{j\geqslant k} D_j(f)(z)\right|^p |\mathrm{d}\mu(z)| = C\int\sum_{k\in\mathbf{Z}} \left|\sum_{j\geqslant k} 2^{(k-j)\alpha}(2^{j\alpha}D_j(f)(z))\right|^p |\mathrm{d}\mu(z)|$$

用 Hölder 不等式 $\left(\frac{1}{p}+\frac{1}{p'}=1\right)$，

$$\mathrm{I} \leqslant C\int\sum_{k\in\mathbf{Z}} \left(\sum_{j\geqslant k} 2^{(k-j)\alpha}\right)^{p/p'} \sum_{j\geqslant k} (2^{(k-j)\alpha} |2^{j\alpha}D_j(f)(z)|^p |d\mu(z)| \leqslant$$

$$C\int\sum_{j\in\mathbf{Z}} |2^{j\alpha}D_j(f)(z)|^p |d\mu(z)| = C\sum_{j\in\mathbf{Z}} (2^{j\alpha}\|D_j(f)\|_p)^p = C\|f\|^p_{B_p^{\alpha,p}}$$

类似地有　　$\mathrm{II} \leqslant C\|f\|^p_{B_p^{\alpha,p}}$

下面估计 III，注意到

$$D_j(f)(z)-D_j(f)(\omega) = \int [D_j(z,\omega)-D_j(\omega,u)] f(u)\mathrm{d}\mu(u) =$$

$$\int [D_j(z,\omega)-D_j(\omega,u)] \sum_l D_l D_l(f)(u)\mathrm{d}\mu(u)$$

记 $E_j(z,\omega,u) = D_j(z,\omega)-D_j(\omega,u)$。容易验证

$$|E_jD_l(z,\omega,v)| \leqslant C2^{-|j-l|}\frac{\rho(z,\omega)}{2^{-j}}\frac{2^{-(j\wedge l)}}{(2^{-(j\wedge l)}+\rho(z,v))^2}$$

因此

$$\mathrm{III} \leqslant C\int\sum_{k\in\mathbf{Z}}\int_{2^{-k}\leqslant\rho(z,\omega)<2^{-k+1}} 2^{k(1+\alpha p)} \left|\sum_{j<k}[D_j(f)(z)-D_j(f)(\omega)]\right|^p |\mathrm{d}\mu(z)||\mathrm{d}\mu(\omega)| \leqslant$$

$$C\int\sum_{k\in\mathbf{Z}}\int_{2^{-k}\leqslant\rho(z,\omega)<2^{-k+1}} 2^{k(1+\alpha p)} \left|\sum_{j<k}\sum_{l\in\mathbf{Z}} E_jD_lD_l(f)(z,\omega)\right|^p |\mathrm{d}\mu(z)||\mathrm{d}\mu(\omega)| \leqslant$$

$$C\int\sum_{k\in\mathbf{Z}}\int_{2^{-k}\leqslant\rho(z,\omega)<2^{-k+1}} 2^{k(1+\alpha p)} \left|\sum_{j<k}\sum_{l<j} |E_jD_lD_l(f)(z,\omega)|\right|^p |\mathrm{d}\mu(z)||\mathrm{d}\mu(\omega)| +$$

$$C\int\sum_{k\in\mathbf{Z}}\int_{2^{-k}\leqslant\rho(z,\omega)<2^{-k+1}} 2^{k(1+\alpha p)} \sum_{j<k}\sum_{l\geqslant j} |E_jD_lD_l(f)(z,\omega)|^p |\mathrm{d}\mu(z)||\mathrm{d}\mu(\omega)| = \mathrm{III}_1+\mathrm{III}_2$$

$$\mathrm{III}_1 \leqslant C\int\sum_{k\in\mathbf{Z}}\int_{2^{-k}\leqslant\rho(z,\omega)<2^{-k+1}} 2^{k(1+\alpha p)} \sum_{j<k}\sum_{l<j}\left[\int |E_jD_l(z,\omega,v)||D_l(f)(v)||\mathrm{d}\mu(v)|\right]^p$$

$$|d\mu(z)||d\mu(\omega)| \leqslant C\int\sum_{k\in\mathbf{Z}}\int_{2^{-k}\leqslant\rho(z,\omega)<2^{-k+1}}2^{k(1+\alpha p)}\sum_{j<k}\sum_{l<j}\int 2^{-(j-l)p}\frac{2^{-(j\wedge l)}}{(2^{-(j\wedge l)}+\rho(z,\omega))^2}$$

$$|D_l(f)(v)|^p|d\mu(v)||d\mu(\omega)||d\mu(z)| \leqslant C\int\sum_{k\in\mathbf{Z}}\sum_{j<k}\sum_{l<j}2^{-(j-l)(1-\alpha)p}|2^{l\alpha}D_l(f)(v)|^p|d\mu(v)| =$$

$$C\int\sum_{j\in\mathbf{Z}}\sum_{l<j}2^{-(j-l)(1-\alpha)p}|2^{l\alpha}D_l(f)(v)|^p|d\mu(v)| = C\int\sum_{l\in\mathbf{Z}}\sum_{j\geqslant l}2^{-(j-l)(1-\alpha)p}|2^{l\alpha}D_l(f)(v)|^p|d\mu(v)| \leqslant$$

$$C\int\sum_{l\in\mathbf{Z}}|2^{l\alpha}D_l(f)(v)|^p|d\mu(v)| = C\sum_{l\in\mathbf{Z}}[2^{l\alpha}\|D_l(f)\|_p]^p = C\|f\|^p_{B_p^{\alpha p}}$$

$$\mathrm{III}_2 \leqslant C\int\sum_{k\in\mathbf{Z}}\int_{2^{-k}\leqslant\rho(z,\omega)<2^{k+1}}2^{k(1+\alpha p)}\sum_{j<k}\sum_{l\geqslant j}\int\left(\frac{\rho(z,\omega)}{2^{-j}}\right)^p\frac{2^{-(j\wedge l)}}{(2^{-(j\wedge l)}+\rho(z,v))^2}|D_l(f)(v)|^p$$

$$|d\mu(v)||d\mu(z)||d\mu(\omega)| \leqslant C\int\sum_{k\in\mathbf{Z}}\sum_{j<k}\sum_{l\geqslant j}2^{(j-l)\alpha p}(2^{l\alpha}|D_l(f)(v)|)^p|d\mu(v)| =$$

$$C\int\sum_{j\in\mathbf{Z}}\sum_{l\geqslant j}2^{(j-l)\alpha p}(2^{l\alpha}|D_l(f)(v)|)^p|d\mu(v)| = C\int\sum_{l\in\mathbf{Z}}\sum_{j<l}2^{(j-l)\alpha p}(2^{l\alpha}|D_l(f)(v)|)^p|d\mu(v)| \leqslant$$

$$C\int\sum_{l\in\mathbf{Z}}(2^{l\alpha}|D_l(f)(v)|)^p|d\mu(v)| = C\sum_{l\in\mathbf{Z}}(2^{l\alpha}\|D_l(f)\|_p)^p = C\|f\|^p_{B_p^{\alpha p}}$$

综合以上估计知

$$\iint\frac{|f(z)-f(\omega)|^p}{\rho(z,\omega)^{1+\alpha p}}|d\mu(z)||d\mu(\omega)| \leqslant C\|f\|^p_{B_p^{\alpha p}} \tag{12}$$

从而定理1得证.

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## A Characterization of Besov Spaces on Spaces of Homogeneous Type

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**Abstract**: A new characterization of the Besov space $B_p^{\alpha p}$ on a space of homogeneous type is given by using an approximation to the identity and Calderorn reproducing formulas.

**Keywords**: homogeneous; approximation to the identity; calderon reproducing formulas; Besov space