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A Note on the Dynamical System Defined on A Self-similar Set Satisfying SSC*

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Abstract: A classical result says, for a self-similar set satisfying the strong separation condition, one can define a continuous self-mapping on it with the normalized Hausdorff measure of the self-similar set as its invariant ergodic measure. A necessary and sufficient condition for the normalized Hausdorff measure to be a measure with maximal entropy for the self-mapping is given.

Key words: self-similar set; discrete dynamics; measure with maximal entropy

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1 The main Problem

Let $D \subset \mathbf{R}^n (n > 0)$ be a closed region and E the self-similar set determined by the similarities $S_j: D \to D$ with the similar ratios $c_j \in (0, 1)$ for $j = 1, 2, \cdots, m$, $m > 1$. Let E satisfy the open set condition, that is, there is a non-empty bounded open set $V \subset \mathbf{R}^n$ such that

$$\bigcup_{j=1}^{m} S_j(V) \subset V$$

$$S_i(V) \cap S_j(V) = \varnothing, 1 \leqslant i < j \leqslant m$$

The Hausdorff dimension $s = \dim_H(E)$ of E is determined by $\sum_{j=1}^{m} c_j^s = 1$ and its s-dimensional Hausdorff measure satisfies $0 < H^s(E) < \infty$.[1] Furthermore, let E satisfy the strong separation condition (SSC), namely,

$$S_i(E) \cap S_j(E) = \varnothing, 1 \leqslant i < j \leqslant m$$

Then we may define a continuous self-mapping

$$\begin{cases} f: E \to E, \\ f(x) = S_j^{-1}(x), x \in S_j(E), j = 1, 2, \cdots, m \end{cases}$$

This is a discrete dynamical system. Using it, we may delve into the more profound fractal behaviors of E, for instance, its properties of Hausdorff measure.

Denote by μ the restriction of $\frac{1}{H^s(E)} H^s$ on E. The restriction of μ on the Borel σ-algebra of E is a probability measure. A classical result says that μ is invariant and ergodic for f, which is independent of the similarity ratios[2].

A remaining problem is whether μ is a measure with maximal entropy for f or not, that is, $h_\mu(f) = \mathrm{ent}(f)$ or not, where the former is the measure-theoretic entropy of f with respect to μ and the latter is the topological entropy of f. By the Variational Principle[3], there holds always $h_\mu(f) \leqslant \mathrm{ent}(f)$. This note solves the problem and the conclusion depends upon the similarity ratios.

2 Main theorem and its Proof

Theorem μ is a measure with maximal entropy for f iff $c_j = c = $ const. for $1 \leqslant j \leqslant m$.

Proof Let $S = \{1, 2, \cdots, m\}$ be the state space with m symbols and $\Sigma_m = \{ i = (i_1 i_2 \cdots i_n \cdots) : i_n \in S, \forall n \geqslant 1\}$, that is, Σ_m is the one sided symbolic space on the state space S. Define

$$\begin{cases} \sigma: \Sigma_m \to \Sigma_m, \\ \sigma(i_1 i_2 \cdots) = (i_2, i_3 \cdots), \ \forall (i_1 i_2 \cdots) \end{cases}$$

namely, σ is the one sided shift. For any $i = (i_1 i_2 \cdots) \in \Sigma_m$, it is easy to see that $\bigcap_{n=1}^{\infty} S_{i_1} S_{i_2} \cdots S_{i_n}(E)$ is a singleton and the unique point contained in it is denoted by $x_i \in E$. By SSC, it is not hard to see that this correspondence is bijective. Define

$$\begin{cases} h: \Sigma_m \to E \\ h(i) = x_i, \ \forall i \in \Sigma_m \end{cases}$$

It is easy to prove that h is a topological conjugacy from σ to f, namely, h is a homeomorphism from Σ_m onto h and there holds $h\sigma = fh$. Hence, their topological entropies are identical, that is, $\mathrm{ent}(f) = \mathrm{ent}(\sigma) = \log m$[3]. It is not hard to see that

$$h(\{ i = (i_1 i_2 \cdots) : i_1 = j\}) =$$

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$$S_j(E),\ j = 1, 2, \cdots, m \qquad (*)$$

and that h induces an affine homeomorphism between $M(\Sigma_m, \sigma)$, $M(E, f)$, which are the sets of the invariant measures of σ, f, respectively:

$$h^*: M(\Sigma_m, \sigma) \to M(E, f)$$

$$h^*(\eta)(A) = \eta(h^{-1}(A)),$$

$$\forall \eta \in M(\Sigma_m, \sigma),\ \forall A \in \mathscr{B}(E)$$

where $\mathscr{B}$ denotes the Borel σ-algebra of E[3,4]. Hence, there is a unique $\eta \in M(\Sigma_m, \sigma)$ such that $h^*(\eta) = \mu = \frac{1}{H^s(E)} H^s$. Take $A = S_j(E)$, by $(*)$ and the definition of h^*, it is easy to see

$$\eta(\{ i = (i_1 i_2 \cdots): i_1 = j\}) =$$

$$\mu(S_j(E)) = c_j^S,\ j = 1, 2, \cdots, m$$

Note that $(c_1^S, c_2^S, \cdots, c_m^S)$ is a probability vector and it is well known that σ is the $(c_1^S, c_2^S, \cdots, c_m^S)$-shift with η[3]. According to (7) in Ref. [3], we have

$$h_\eta(\sigma) = -\sum_{j=1}^{m} c_j^S \cdot \log c_j^S$$

By Theorem 4.2 and Corollary 4.2.1 in Ref. [3], $h_\eta(\sigma) = \log m$ iff $c_j = c = \text{const.}$ for $1 \leqslant j \leqslant m$.

Furthermore, it is not difficult to see that the homeomorphism $h: \Sigma_m \to \Sigma$ induces an isomorphism from the probability space $(\Sigma_m, \mathscr{B}(\Sigma_m), \eta)$ to the probability space $(E, \mathscr{B}(E), \mu)$[3]. By Ref. [3, theorem 4.11], we have $h_\eta(\sigma) = h_\mu(f)$.

Finally, summing up the above, we get that $h_\mu(f) = \text{ent}(f) = \log m$ iff $c_j = c = \text{const.}$, for $1 \leqslant j \leqslant m$. The problem is done.

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关于定义在满足强分离条件的自相似集上的动力系统的一点注记

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摘 要：在满足强分离条件的自相似集上，可以定义一个连续自映射。这个自相似集的单位化的 Hausdorff 测度是它的不变遍历测度。还给出这个遍历测度是该自映射的一个极大熵测度的充要条件。

关键词：自相似集；离散动力系统；极大熵测度

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