On Periodic Solutions of First-Order Linear and Nonlinear Functional Differential Equations*

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Abstract: This paper is concerned with the periodic solutions of first-order linear and nonlinear functional differential equations. Some sufficient conditions of the existence and uniqueness of the periodic solutions are obtained. It improves some well-known results.

Key words: linear and nonlinear; functional differential equation; infinite deviating arguments; periodic solutions

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1 Introduction

Recently, the periodic solutions of functional differential equations have been widely studied[1-6]. Reference[1] considered the functional differential equation

$$\dot{x}(t) = b(t, x(t+\cdot)) + G(t, x(t+\cdot)) \tag{1}$$

Where $x(t) \in R^n$, $x(t+\cdot) \in BC(R, R^n)$ is given by $x(t+\cdot)(s) = x(t+s)$, b and G are continuous and boundary mappings from $R \times BC(R, R^n)$ to R^n, for every fixed $t \in R$, $b(t, \varphi)$ is linear with respect to $\varphi \in BC(R, R^n)$. There exists a constant $T > 0$, such that

$$b(t+T, \varphi) = b(t, \varphi), G(t+T, \varphi) = G(t, \varphi)$$

Moreover,

$$\lim_{\|\varphi\| \to \infty} \frac{|G(t, \varphi)|}{\|\varphi\|} = 0$$

is uniform for $t \in R$, where $|\cdot|$ and $\|\cdot\|$ denote the norms in R^n and $BC(R, R^n)$ respectively. The main result is that if the linear equation $\dot{x}(t) = b(t, x(t+\cdot))$ has only a trivial T-periodic solution, then Eq (1) has at least one T-periodic solution. Particularly, if $G(t, \varphi) = G(t)$, it is easy to show that Eq (1) has a unique T-periodic solution. Reference [2] studied the functional differential equation

$$u'(t) = l(u)(t) + g(t) \tag{2}$$

where $l: C_w(R)$ is a linear bounded operator and $g \in L_w(R)$, and the important particular case

$$u'(t) = \sum_{k=1}^{n} P_k^{(t)} u(\tau(t)) + g(t) \tag{3}$$

Some sufficient conditions are established for the existence of unique T-periodic solutions of Eq (2) and Eq (3).

In the present paper, we consider the linear equation with infinite deviating arguments

$$\dot{x}(t) = p(t)x(t - \tau(t)) + g(t) \tag{4}$$

and the nonlinear equation with infinite deviating arguments

$$\dot{x}(t) = p(t) f(x(t - \tau(t))) \tag{5}$$

Some new optimal sufficient conditions are established for the existence of a unique T-periodic solution of Eq (4) and the trivial T-periodic solution of Eq (5).

2 Main Results and Proof

Theorem 1　Assume that $p(t)\tau(t), g(t) \in C(R, R^n)$, $p(t) \geqslant 0$, There exists a constant $T > 0$, such that

$$p(t+T) = p(t), \tau(t+T) = \tau(t),$$
$$g(t+T) = g(t)$$

If

$$\int_0^T p(t)\,dt < 4 \tag{6}$$

then Eq (4) has a unique T-periodic solution.

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Proof According to reference [1], it is sufficient to show that the corresponding homogeneous equation

$$\dot{x}(t) = p(t)x(t-\tau(t)) \tag{7}$$

has only a trivial T-periodic solution. Assume the contrary. Let there exist a nontrivial T-periodic solution $x(t)$ of Eq (7), then $\max_{0\leqslant t\leqslant T} |x(t)| > 0$, noticing that Eq (7) is linear, without loss of generality, put

$$M = \max_{0\leqslant t\leqslant T} |x(t)| > 0$$

There are the following two possible cases:

Case 1

$$\max_{0\leqslant t\leqslant T} |x(t)| \geqslant 0$$

then $x(t) \geqslant 0$, for all $t \geqslant 0$, since $p(t) \geqslant 0$, hence $\dot{x}(t) \geqslant 0$. Which is a contradiction to the assumption.

Case 2 $\max_{0\leqslant t\leqslant T} |x(t)| < 0$. Choose $t^* \in [t_*, t_* + T]$, such that $x(t_*) = \max_{0\leqslant t\leqslant T} x(t) = -m$, $x(t^*) = \max_{0\leqslant t\leqslant T} x(t) = M$.

Integrating Eq (7) from t_* to t^* and from t^* to $t_* + T$, we obtain

$$M + m = \int_{t_*}^{t^*} p(t)x(t-\tau(t))\,dt \leqslant M\int_{t_*}^{t^*} p(t)\,dt \tag{8}$$

and

$$m + M = -\int_{t^*}^{t_*+T} p(t)\,x(t-\tau(t))\,dt \leqslant m\int_{t^*}^{t_*+T} p(t)\,dt \tag{9}$$

Summing the last two inequalities:

$$4 \leqslant 2 + \frac{m}{M} + \frac{M}{m} \leqslant \int_{t_*}^{t_*+T} p(t)\,dt = \int_0^T p(t)\,dt$$

which is a contradiction to the condition (6). The proof is complete.

Theorem 2 Assume that $p(t), \tau(t) \in C(R, R^n)$, $p(t) \geqslant 0$, There exists a constant $T > 0$, such that $p(t+T) = p(t)$, $\tau(t+T) = \tau(t)$, $f(x)$ is continuous and $xf(x) > 0\ (x \neq 0)$. If $\frac{f(x)}{x} \leqslant 1$ and

$$\int_0^T p(t)\,dt < 4$$

then Eq (5) has only a trivial T-periodic solution.

Proof Assume the contrary. Let there exist a nontrivial T-periodic solution $x(t)$ of Eq (5), then $\max_{0\leqslant t\leqslant T} |x(t)| > 0$. There are also the following two possible cases:

Case 1 $\max_{0\leqslant t\leqslant T} x(t) < 0$, then $x(t) < 0$, since $p(t) \geqslant 0$, so $\dot{x}(t) \leqslant 0$, which is a contradiction to the assumption.

Case 2 $\max_{0\leqslant t\leqslant T} x(t) \geqslant 0$, the proof of this part is similar to Theorem 1, so it is omitted here.

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一阶线性与非线性泛函微分方程的周期解

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摘　要：研究了一阶非线性泛函微分方程的周期解，得到了周期解存在唯一的一些充分条件，推广和改进了已有的结果。

关键词：非线性泛函微分方程；周期解；无界偏差

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