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Study of the Gravitational Energy in Neutron Stars *

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Abstract: The gravitational field in a neutron star is discussed in the general static isotropic metric. Different from previous works, the contribution of gravitational field to the mass of neutron star is thought to be due to the invariant volume element, and so the definition of mass of neutron star is clerified.

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The neutron star is formed due to gravitational collapse, and it's many properties are decided by gravitational field which has been studied extensively[1-4] during previous several decades. The mass of neutron star is the first problem to which we face in the study of neutron star. In general, the definitions of the mass of neutron star are classified into gravitational mass which includes the energy of both gravitational field and pure neutron star matter, baryonic mass which is the sum of free mass of all baryons in a neutron star, and mass of matter which is only the energy of pure matter in a neutron star. These definitions are very clear, but there still exist some degrees of misunderstanding in the calculation of these masses. The stability and structure of neutron star in which gravity shares an key role is an importantproblem in the neutron star. To a static spherical symmetrical neutron star, it' s gravitational structure is relatively simple which can be described by TOV equation deduced from Einstein equation[1]. To a rotating neutron star, it's theoretical foundation is still Einstein equation, however, due to variousfactors such as oscillation[5-7], rotatoin[8,9] etc, it's gravitational property is very complex. Especially, in the evolution of neutron star, the gravitational radiation causes secular instability[10] which may result in the dynamical instability finally. At all, the gravitational property of neutron star is an important aspect in the study of neutron star.

The direct calculation of gravitational energy in neutron star is difficult, so we usually calculate the total energy of neutron star and analyze it's gravitational energy in general relativity. In this paper, we pay attention to the following problem, that is why the integral to the space coordinates of internal total energy density of neutron star matter which doesn't include the contribution of gravity includes gravitational energy density. This is an old and important problem, Weinberg call it a phony error, and the explanation to it is fuzzy so far. We explain it by invariant volume element in general relativity, and the contribution of gravity is understood in this integral.

The next of this paper is as follows: In the second section, we will list the source of problem, and we present our explanation in the third section, finally, we give a summary to this problem.

1　The source of problem

In the cold static neutron star, the structure is decided by both the internal property of neutron star matter and the distributing of the gravitational field of neutron star, and they are treated by hydrodynamical and general relativity. Here, we concern it's gravitational property mainly. To make the problem simple, we assume the neutron star is static and spherical symmetry. And the metric is as following:

$$
\begin{aligned}
g_{rr} &= A(r), \\
g_{\theta\theta} &= r^2, \\
g_{\varphi\varphi} &= r^2 \sin^2 \theta, \\
g_{tt} &= -B(r) \\
g_{\mu\nu} &= 0, \text{ as } \mu \neq \nu
\end{aligned}
\tag{1}
$$

Assume the energy-momentum tensor of neutron star

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matter is the same as the one of ideal liquid, that is:

$$T^{\mu\nu}=pg^{\mu\nu}+(p+\rho)U^{\mu}U^{\nu}\tag{2}$$

here p, ρ are the pressure and total energy density of neutron star matterrespectively, and U^{μ} is 4-dimensional vector, which is defined as:

$$g^{\mu\nu}U_{\mu}U_{\nu}=-1\tag{3}$$

From Einsteinequation, we can obtain the structure equation of static sphericalsymmetrical neutron star (also called TOV equation) as following:

$$-r^2p'(r)=GM(r)\rho(r)\left[1+\frac{p(r)}{\rho(r)}\right]\cdot\left[1+\frac{4\pi r^2p(r)}{M(r)}\right]\left[1-\frac{2GM(r)}{r}\right]^{-1}\tag{4}$$

in which

$$M'(r)=4\pi r^2\rho(r)\tag{5}$$

Integrating to (5) in the range of whole neutron star, and the mass of neutron star is obtained:

$$M(R)=\int_0^R4\pi r^2\rho(r)\mathrm{d}r\tag{6}$$

In the left of (6), gravitational energy is included. To understand, we quote the proving of Weinberg. Let's discuss it in the quasi-Minkowsiki coordinate, the metric $g_{\mu\nu}$ of which access to Minkosiki metric $g_{\mu\nu}$ as very far from the studied finite system of matter. Assuming

$$g_{\mu\nu}=h_{\mu\nu}+\eta_{\mu\nu}\tag{7}$$

so that $h_{\mu\nu}$ access to zero in the very far place. The total energy and momentum of the system is

$$P^{\lambda}=-\frac{1}{8\pi G}\int Q^{j_0\lambda}n_ir^2\mathrm{d}\Omega\tag{8}$$

it is integrated on the spherical surface of which radius is r, $\boldsymbol{n}$ is the outward vertical vector, $\mathrm{d}\Omega$ is the three-dimensional angle element, that is

$$r\equiv(x_ix_i)^{1/2}\quad n_i=\frac{x_i}{r}$$

$$\mathrm{d}\Omega=\sin\theta\mathrm{d}\theta\mathrm{d}\varphi\tag{9}$$

$$Q^{\rho\nu\lambda}=\frac{1}{2}\left\{\frac{\partial h^{\mu}_{\mu}}{\partial x^{\nu}}\eta^{\rho\lambda}-\frac{\partial h^{\mu}_{\mu}}{\partial x_{\rho}}\eta^{\nu\lambda}-\frac{\partial h^{\mu\nu}}{\partial x^{\mu}}\eta^{\rho\lambda}+\frac{\partial h^{\mu\rho}}{\partial x^{\mu}}\eta^{\nu\lambda}+\frac{h^{\nu\lambda}}{\partial x_{\rho}}-\frac{h^{\rho\lambda}}{\partial x_{\nu}}\right\}\tag{10}$$

So the total energy is

$$P^0=-\frac{1}{16\pi G}\int\left\{\frac{\partial h_{jj}}{\partial x^i}-\frac{\partial h_{ij}}{\partial x^j}\right\}n_ir^2\mathrm{d}\Omega\tag{11}$$

As $r\to\infty$,

$$h_{ij}\equiv g_{ij}-\delta_{ij}\to\frac{2MG}{r}n_in_j+o\left(\frac{1}{r^2}\right)\tag{12}$$

using

$$\frac{\partial r}{\partial x^i}=n_i\qquad\frac{\partial n_i}{\partial x^j}=\frac{\delta_{ij}-n_in_j}{r}\tag{13}$$

to obtain

$$\frac{\partial h_{jj}}{\partial x^i}-\frac{\partial h_{ij}}{\partial x^j}\to-\frac{4MG}{r^2}n_i+o\left(\frac{1}{r^3}\right)\tag{14}$$

Replace the difference in (11) with (14)

$$P^0=M\tag{15}$$

So M includes gravitational energy ρ_0. But as we know, ρ in the right of (6) is the internal energy of neutron star matter which do not include the gravitational energy. The energy density which do not include the gravitational energy being integrated to the mass which include the gravitational energy, it seems to be a contradiction which Weinberg called a phony error.

2 The solution to the problem

Weinberg thought that the total energy of matter is not determinate, and M hasn't been defined to be the total energy of pure matter. The difficulty of this solution is that we can not know M include the gravitational energy similarly. We resolve it as following, in the following figure, we describe the same volume element observed by the observer in the local inertial frames and the distant observer respectively.

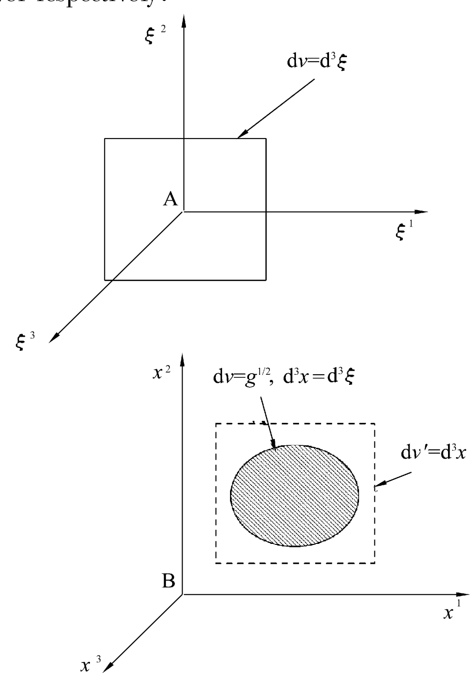

Fig. 1　The same volume cell in two different coordinates

To A, the observer in the local inertial frames, the volume element (the solid rectangle) is $\mathrm{d}v=\mathrm{d}^3\zeta$, in which the influence of the gravity do not exist according to general relativity. The same volume element to B, the distant observer, should be bent due to the gravity as the hatching. So the samevolume element $\mathrm{d}v$ is $\mathrm{d}^3\zeta$ to the observer A and is $\sqrt{g}\mathrm{d}^3x$ to the observer B, that is,

$$\mathrm{d}^3\zeta=\sqrt{g}\mathrm{d}^3x\tag{16}$$

It is so called invariant volume element. Multiplying the two sides of equation (16) by the internal energy density of neutron star matter ρ and integrating to the whole space, we can obtain the total energy of pure neutron star matter,

$$M_{\text{matter}} = \int \rho d^3 \zeta = \int \rho \sqrt{g} d^3 x \qquad (17)$$

If we multiply the internal energy density ρ by the volume element dv' (dashed rectangle), the energy $\rho dv'$ will be larger than the energy $\rho d^3 \zeta$. This difference results from the volume element bent by the gravity to B and $\rho dv'$ still treating dv' as dv, the volume when the gravity does not exist. $d^3 x - \sqrt{g} d^3 x$ is the part of the volume element bent by gravity, so the excessive energy $\rho d^3 x - \sqrt{g} d^3 x)$ in $\rho d^3 x$ is due to the influence of the gravity which we can call the gravitational energy in $dv' = d^3 x$. So we can say

$$M = \int \rho d^3 x \qquad (18)$$

that is (6), is the total energy of matter and gravitational field.

The coordinate in (6) is as B in the above figure, becausethe definition of (6) is a part of (4) which describes the gravitational structure of static stars.

3 Summary

In this paper, we try to describe the definition of neutron star's mass clearly, it is the foundation of study of the neutron star. And the contribution of gravity in the total mass of neutron star is due to the change of the integral range in local and general coordinates.

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中子星内引力能的研究

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摘 要：用一般静态各向同性度规研究了中子星的引力能。认为引力场对中子星质量的贡献是由于不变体积元，从而使中子星质量的定义得到澄清。

关键词：引力场；中子星；质量

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