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Topological Structures of Teichmüller Spaces*

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Abstract: Several topological structures in finite dimensional Teichmüller spaces are studied. And it's proved that some topological structures in finite dimensional Teichmüller spaces are topologically equivalent. A metric defined by the length spectrum of Riemann surface is given in infinite dimensionalreduced Teichmüller space.

Key words: Teichmüller space; Riemann surface; topological structure
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1 Introduction

Let X be a Riemann surface of hyperbolic type. For any Riemann surface X_0 of the same topological type as X and any quasiconformal mapping $f_0: X \to X_0$, we denote the pair (X_0, f_0) as a marked Riemann surface. Two marked Riemann surfaces (X_1, f_1) and (X_2, f_2) are equivalent if there is a conformal mapping $\phi: X_1 \to X_2$ which is homotopic to $f_2 \circ f_1^{-1}$. Denote $[X_0, f_0]$ to be the equivalent class of (X_0, f_0). The Teichmüller space $T(X)$ is the set of the equivalent classes.

As we know, Teichmüller[1] gave a complete metric on $T(X)$ using the dilatation of extremal quasiconformal mapping:

$$d_T([X_1, f_1], [X_2, f_2]) = \log K[f_0]$$

where $f_0: X_1 \to X_2$ is the extremal quasiconformal mapping in the homotopic class of $f_2 \circ f_1^{-1}$, and $K[f_0]$ is its dilatation.

For topological finite Riemann surface X, the Teichmüller metric between two points is equal to the log of the superum of the ratios of the extremal lengths of simple closed curves on the corresponding Riemann surfaces. The Teichmüller metric d_T gives a topological structure on $T(X)$. We'll give a topological structure on $T(X)$ by the extremal length functions of simple closed curves. And we'll show that this topological structure is topologically equivalent to the one given by the Teichmüller metric. By the Fenchel-Nielsen coordinates of $T(X)$, there exist a topological structure on $T(X)$ given by the hyperbolic length functions of simple closed curves. This topological structure will be shown to be topologically equivalent to a topological structure defined by a metric, the length spectrum metric[2,3] which defined by the log of superum of the ratios of hyperbolic lengths of simple closed curves on the corresponding Riemann surfaces. Combing with some of our previous results[4], we'll show that these topological structures are topologically equivalent to each other.

For topological infinite Riemann surface X, we'll prove that the length spectrum metric still exist. This metric defines a topological structure on $T(X)$. Almost by the same way we'll show that the hyperbolic length functions of simple closed curves define a topological structure on $T(X)$. And these two topological structures are topologically equivalent.

2 Topological finite Riemann surfaces

The Riemann surfaces in this section are of topological finite type and of non-elementary type. Let $\Sigma_X = \{\gamma_i\}$ be the set of the representatives of elements (not including the unit element) of the fundamental group $\pi_1(X)$ of surface X. Let $l_X(\gamma)$ be the shortest length under the Poincaré metric in the freely homotopic class of closed curve γ on the Riemann surface X. $l_X(\gamma)$ is also called the Poincare length (or hyperbolic length) of γ. The sequence $l_X(\gamma_j)$ corresponding to Σ_X is called the length spectrum of Riemann surface X.[1] Let Σ'_X be the set of homotopic classes of curves in Σ_X which don't homotopic to a simple puncture and Σ''_X be the set of homotopic classes of simple closed curves in Σ'_X. We

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introduce a metric on $T(X)$ using length spectrum of Riemann surface[2, 3]:

$$d([X_1, f_1], [X_2, f_2]) = \log \rho([X_1, f_1], [X_2, f_2]) \quad (1)$$

where

$$\rho([X_1, f_1], [X_2, f_2]) = \sup_{\gamma\in\Sigma_{X_1}} \left\{ \frac{l_{X_2}(f(\gamma))}{l_{X_1}(\gamma)}, \frac{l_{X_1}(\gamma)}{l_{X_2}(f(\gamma))} \right\}$$

where $f = f_2 \circ f_1^{-1}$. This metric is called the length spectrum metric of the Teichmüller space $T(X)$. And by the results of Thurston[5, 6], we know that the length spectrum metric can be characterized only by the log of ratio of hyperbolic lengths of simple closed curves on the corresponding Riemann surfaces. That is:

$$d([X_1, f_1], [X_2, f_2]) = \log \rho([X_1, f_1], [X_2, f_2]),$$

where

$$\rho([X_1, f_1], [X_2, f_2]) = \sup_{\gamma\in\Sigma_{X_1}} \left\{ \frac{l_{X_2}(f(\gamma))}{l_{X_1}(\gamma)}, \frac{l_{X_1}(\gamma)}{l_{X_2}(f(\gamma))} \right\},$$

$$f = f_2 \circ f_1^{-1}$$

For any $\alpha \in \Sigma''_X$, let $E_\alpha(X)$ be the extremal length of α on X.

The following is a natural generalization of a result of Kerckhoff[4, 7, 8].

Lemma 1 For any two points $[X_1, f_1]$ and $[X_2, f_2]$ in $T(X)$, we have

$$d_T([X_1, f_1], [X_2, f_2]) = \frac{1}{2}\log \sup_{\alpha\in\Sigma''_{X_1}} \frac{E_\alpha(X_1)}{E_{f(\alpha)}(X_2)}$$

where $f = f_2 \circ f_1^{-1}$, Σ''_{X_1} is the set of homotopic classes of simple closed curves in Σ'_{X_1}.

Remark This result for compact Riemann surface was obtained by Kerckhoff[7]. Because the proof of the above result is the same as that of Kerckhoff, we omit the details. On the other hand, we don't know for any topological infinite type Riemann surface whether the above result remains valid.

By Lemma 1, we have the following embedding[9]

$$\Phi: T(X) \to R^{+}_{\Sigma''_X} \quad (2)$$

with

$$\Phi([X_0, f_0]) = \left\{ E_{f_0(\alpha)}(X_0) \right\}_{\alpha\in\Sigma''_X}$$

for any $[X_0, f_0] \in T(X)$

Gardiner and Masur[9] gave the above embedding for Teichmüller spaces of conformal finite Riemann surfaces. By Lemma 1, we know that the above embedding is still valid for Teichmüller spaces of topological finite Riemann surfaces.

Next we introduce a result of Maskit[10].

Lemma 2 For any $\gamma \in \Sigma''_X$, We have the following inequality

$$\frac{l_X(\gamma)}{\pi} \leqslant E_\gamma(X) \leqslant \frac{1}{2} l_X(\gamma)\exp\left(\frac{l_X(\gamma)}{2}\right)$$

The metric d_T define a topological structure on $T(X)$ and we call it d_T-structure and the metric d define another topological structure on $T(X)$ and we call it d-structure. In [4] we proved that the metric d_T is topologically equivalent to d on $T(X)$. So the d_T-structure is topologically equivalent to the d-structure on $T(X)$.

For any $\alpha \in \Sigma''_X$, $E_{f_0(\alpha)}([X_0, f_0])$ defines a function on $T(X)$ as $[X_0, f_0]$ varies on $T(X)$. Let $\Pi = E_{f_0(\alpha)}([X_0, f_0])$, $\alpha \in \Sigma''_X$ be the set of extremal length functions defined on $T(X)$.

Theorem 1 The collection $\{f^{-1}(U) \mid U$ open in $\mathbf{R}, f \in \Pi\}$ forms a subbasis for a topological structure (E-structure) of $T(X)$. The E-structure is topologically equivalent to the d_T-structure on $T(X)$.

Proof Because the extremal length functions are continuous on $T(X)$[11]. It's easy to check that the collection $\{f^{-1}(U) \mid U$ open in $\mathbf{R}$, $f \in \Pi\}$ forms a subbasis for a topological structure (the E-structure) of $T(X)$.

To prove the E-structure is topologically equivalent to the d_T-structure, it's sufficient to prove the following: For any sequence of points $[X_n, f_n]$, $n = 1, 2, \cdots$ in $T(X)$, they converge in the d_T-structure if and only if they converge in the E-structure.

First let the sequence of points $[X_n, f_n]$, $n = 1, 2, \cdots$ converges in the d_T-structure to some point $[X_0, f_0]$. Then by Lemma 1 and the embedding (2), it's easy to know that the sequence of points $[X_n, f_n]$, $n = 1, 2, \cdots$ converges in the E-structure to the same point $[X_0, f_0]$.

Secondly let the sequence of points $[X_n, f_n]$, $n = 1, 2, \cdots$ converges in the E-structure to some point $[X_0, f_0]$. Then for any $\varepsilon > 0$, there exists some N, such that for any $n \geqslant N$ and any $\alpha \in \Sigma''_X$,

$$\left| E_{f_n(\alpha)}(X_n) - E_{f_0(\alpha)}(X_0) \right| < \varepsilon$$

Because X_0 is a Riemann surface of topological finite type, by the well-known compactness theorem, there exists positive number B, such that for any $\alpha \in \Sigma''_{X_0}$, $l_{X_0}(\alpha) \geqslant B$. By Lemma 1 and Lemma 2, for any $\alpha \in \Sigma''_{X_0}$, $E_\alpha(X_0) \geqslant B/\pi$. So for $n \geqslant N$, and for any $\alpha \in \Sigma''_X$,

$$\left| \frac{E_{f_n(\alpha)}(X_n)}{E_{f_0(\alpha)}(X_0)} - 1 \right| < \frac{\varepsilon\pi}{B}$$

This implies that the sequence of points $[X_n, f_n]$, $n = 1, 2, \cdots$ converges in the d_T-structure to the same point $[X_0, f_0]$.

For any $\alpha \in \Sigma''_X$, $L_{f_0(\alpha)}([X_0, f_0]) = l_{X_0}(f_0(\alpha))$ defines a function on $T(X)$ as $[f_0, X_0]$ varies on $T(X)$. Let $\Psi = L_{f_0(\alpha)}([X_0, f_0])$, $\alpha \in \Sigma''_X$ be the set of hyperbolic length functions defined on $T(X)$. Then the collection $\{f^{-1}(U) \mid U \text{ open in } \mathbf{R},\ f \in \Psi\}$ forms a subbasis for a topological structure of $T(X)$. This result can be obtained directly from the Fenchel-Nielsen coordinates of Teichmüller spaces of finite topological type Riemann surfaces. We call this topological structure H-structure.

Similar to the proof of Theorem 1, we have:

Theorem 2　The d-structure is topologically equivalent to the H-structure on $T(X)$.

By [4], the d-structure is topologically equivalent to the d_T-structure on $T(X)$. We have:

Corollary 1　The E-structure, the H-structure, the d-structure and the d_T-structure on $T(X)$ are topologically equivalent to each other.

3　Topological infinite Riemann surfaces

The Riemann surfaces in this section are of topological infinite type. Just as the topological finite cases, we define:

$$d([X_1, f_1], [X_2, f_2]) = \log \rho([X_1, f_1], [X_2, f_2]) \tag{3}$$

where

$$\rho([X_1, f_1], [X_2, f_2]) = \sup_{\gamma \in \Sigma_{X_1}} \left\{ \frac{l_{X_2}(f(\gamma))}{l_{X_1}(\gamma)}, \frac{l_{X_1}(\gamma)}{l_{X_2}(f(\gamma))} \right\}$$

where $f = f_2 \circ f_1^{-1}$. This defines a function on $T(X)$. From the proof of the relevant results of Thurston[5,6], we know that, for topological infinite Riemann surfaces, the function (3) still can be characterized only by the log of ratios of hyperbolic lengths of simple closed curves on the corresponding Riemann surfaces. That is:

$$d([X_1, f_1], [X_2, f_2]) = \log \rho([X_1, f_1], [X_2, f_2])$$

where

$$\rho([X_1, f_1], [X_2, f_2]) = \sup_{\gamma \in \Sigma''_{X_1}} \left\{ \frac{l_{X_2}(f(\gamma))}{l_{X_1}(\gamma)}, \frac{l_{X_1}(\gamma)}{l_{X_2}(f(\gamma))} \right\}$$

$$f = f_2 \circ f_1^{-1}$$

We'll show that $d(\circ, \circ)$ is also a metric for the Teichmüller space of topological infinite Riemann surfaces.

Theorem 3　$d(\circ, \circ)$ is a metric in the Teichmüller space $T(X)$ of topological infinite type Riemann surface X.

Proof　To prove that $d(z_1, z_2)$ is a metric, it's sufficient to prove

(a) $d(z_1, z_2) \geqslant 0$, for any $z_1, z_2 \in T(X)$, and $d(z_1, z_2) = 0$ if and only if $z_1 = z_2$.

(b) $d(z_1, z_2) = d(z_2, z_1)$, for any $z_1, z_2 \in T(x)$.

(c) $d(z_1, z_3) \leqslant d(z_1, z_2) + d(z_2, z_3)$ for any $z_1, z_2, z_3 \in T(X)$.

From the definition of $d(\circ, \circ)$, (b) and (c) are obvious. We only need to prove (a). By the definition again we know that for any $z_1, z_2 \in T(x)$, $d(z_1, z_2) \geqslant 0$.

To prove the theorem, we only need to prove that $d(z_1, z_2) = 0$ if and only if $z_1 = z_2$. It's obvious that $z_1 = z_2 \Rightarrow d(z_1, z_2) = 0$. We only need to prove that $d(z_1, z_2) = 0$ implies $z_1 = z_2$. Let $z_1 = [X_1, f_1]$ and $z_2 = [X_2, f_2]$. It's sufficient to find an isometry (or conformal mapping) between z_1 and z_2 which is homotopic to $f_2 \circ f_1^{-1}$ (we called it preserving the marking for the sake of convenience).

In the following discussion, we'll not distinguish between the homotopic class of closed curves and the geodesic in this homotopic class. Take a sequence of disjoint, distinct elements $\alpha_j \in \Sigma''_X$, $j = 1, 2, \cdots$, so that after cutting the Riemann surface X along the geodesics $\alpha_j \in \Sigma''_X$, $j = 1, 2, \cdots$, the Riemann surface X is divided into a sequence of Riemann surfaces X_j, $j = 1, 2, \cdots$ and each X_j, $j = 1, 2, \cdots$ is a Riemann surface of non-elementary topological finite type with geodesic boundaries. By the same way, we cut X_i, $i = 1, 2$ along the geodesic $f_i(\alpha_j)$, $i = 1, 2$, $j = 1, 2, \cdots$ and X_i, $i = 1, 2$ are divided into X_i^j, $i = 1, 2$, $j = 1, 2, \cdots$ and each of it is a Riemann surface of non-elementary topological finite type with geodesic boundaries.

It is well-known that any Riemann surface with geodesic boundaries is the unique Nielsen kernel of a Riemann surface with complete hyperbolic metric. Consider now X_i^j, $i = 1, 2$, $j = 1, 2, \cdots$. They are Nielsen kernels of Riemann surfaces X_i^j, $i = 1, 2$, $j = 1, 2, \cdots$. Because $d(\circ, \circ)$ is a metric for Teichmüller spaces of topological finite Riemann surfaces, and together by (3), $d(X_1^j, X_2^j) = 0$, $j = 1, 2, \cdots$. That is X_1^j, $j = 1, 2, \cdots$ is isometric to X_2^j, $j = 1, 2, \cdots$ and preserves the marking. So each part of X_1 is isometric to the corresponding part of X_2 and preserves the marking. To get an isometry between X_1 and X_2 which preserves the marking, the remaining work that we should do is to know whether the gluing types of the cutting curves $f_1(\alpha_j)$ and $f_2(\alpha_j)$ are the same, $j = 1, 2, \cdots$. Take any pair of curves $f_1(\alpha_j)$ and $f_2(\alpha_j)$. Consider a connecting union X_1^* of finite number of X_1^j which contains $f_1(\alpha_j)$ with

$f_1(\alpha_j)$ is not a boundary geodesic. And consider the corresponding part X_2^* on X_2. Repeat the previous discussion, we know by (3) that $d(X_1^*, X_2^*) = 0$. Because X_1^* and X_2^* are of topological finite type, X_1^* is isometric to X_2^* and preserves the marking. This implies that the gluing types of the cutting curves $f_1(\alpha_j)$ and $f_2(\alpha_j)$ are the same. Because $f_1(\alpha_j)$ and $f_2(\alpha_j)$ are arbitrary, X_1 is isometric to X_2 and preserves the marking. This means $[X_1, f_1] = [X_2, f_2]$.

By the same way as in section 2, we can define the d-structure, the d_T-structure and the H-structure on $T(X)$ of the topological infinite type.

The same as the proof of the Theorem 3, we have,

Corollary 2 The d-structure and the H-structure on Teichmüller space of Riemann surfaces of topological infinite type are topologically equivalent to each other.

It's natural to ask the following problem: Is the d-structure topologically equivalent to the d_T-structure on the Teichmüller spaces of topological infinite type Riemann surfaces.

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Teichmüller 空间的拓扑结构

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摘　要：研究了有限维 Teichmüller 空间上的拓扑结构，证明了有限维 Teichmüller 空间中一些拓扑结构的相互拓扑等价性。证明了可以利用黎曼曲面的长度谱定义无穷维 Teichmüller 空间上的一个度量。

关键词：Teichmüller 空间；黎曼面；拓扑结构

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Sturm Comparison Theorems for Second Order Neutral Nonlinear Differential Equations

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Abstract: Some differential inequalities are established, and the oscillation for the second order neutral nonlinear differential equations are discussed and some Sturm comparison theorems are obtained. The results generalize some classical Sturm comparison theorems.

Key words: neutral differential equation; differential inequality; oscillation; Sturm comparison theorem