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The Octet Periodic Law of Natural Numbers and the Four Theorems of Diophantus*

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Abstract: On the basis of discovering the Octet Periodic Law of natural numbers, the four theorems of Diophantus are formulated. This solves exhaustively the old problem of decomposing any positive integer into a sum of squares.

Key words: periodic table; Octet Periodic Law; ocete operator; embryonic number; discriminant; discriminator; four theorems of Diophantus

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1 The Octer Periodic Law of Natural Numbers

We denote the set of natural numbers {1, 2, 3, …} by **N**. Thus, $0 \notin \mathbf{N}$. While we are travelling along the chain of successors by starting from 1, we feel inwardly that the properties of natural numbers pulsate at a pace of eight consecutive integers. Thus, drawing inspiration from the Periodic Law of Chemical Elements, we discovered the Octet Periodic Law of Natural Numbers. Each such an eight consecutive integers forms a recurring period, the elements of which belong to the groups $8n-k$ ($k=7, 6, \cdots, 1, 0$), which are designated as the groups A, B, C, D, E, F, G, and H respectively. The symbols $A, B, \cdots, H$ are called specifically octet codes.

Arranging the elements of N in this way, it is recognized that, there is a periodic tendency towards recurrence of similar properties, thereby producing the octet periodic table of natural numbers, as shown in table 1. In the table, the prime numbers are shown in boldface. All the representative elements in any given vertical column in the table have the same representation as a sum (of the least number) of squares. Elements in the same row of the table are said to be in the same period. Thus this building-up scheme brings to natural numbers the most naturel principles of classification and the most natural hierarchy of method. The fundamental idea expressed by these classifications comes to be known as the ***Octet Periodic Law of Natural numbers***: ***The mathematical properties of the positive integers are periodic functions of their octet codes.***

2 The Manipulation of Octet Codes

Consider the finite set τ of the octet codes $\{A, B, C, D, E, F, G, H\} \equiv \tau$. Using trivial logical operation, it is easy to work out the algorithmic expression

$$C \circ G = E$$

i.e., the product of an element of group C by an element of group G transforms into an element of group E.

In the same way, we can have $B \circ D = H$, $A \circ C = C$, $E \circ E = A$, etc.

Fig. 1 gives the mathematical logic system of the multiplication operation of the octet codes by letting $x \circ y$ be the entry in row x and column y.

Multipliers

Multiplicands	A	B	C	D	E	F	G	H
A	A	B	C	D	E	F	G	H
B	B	D	F	H	B	D	F	H
C	C	F	A	D	G	B	E	H
D	D	H	D	H	D	H	D	H
E	E	B	G	D	A	F	C	H
F	F	D	B	H	F	D	B	H
G	G	F	E	D	C	B	A	H
H	H	H	H	H	H	H	H	H

Fig 1　Multiplication of octet codes

We see from above that the octet codes $A, B, \cdots, H$ possess both the properties of numbers and operators, so it is convenient to call them octet operators among

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which we are particularly interested in the odd operators A, C, E, and G which are responsible to the representation of any positive integer as a sum of squares.

By convention, we call C, G, the "hard" operators because of their eccentric and egotistic characters, and A, E the "soft" operators because of their affinity tending to combine with the operators they acted on. Furthermore, owing to the "hardness" of C, G, we do transform neither CG into E, nor C^2, G^2, or C^2, C^2G^2, G^2 into A.

Using Fig. 1, it is readily to evaluate the powers of the octet operators and the results are listed in Table 2.

Tab. 1 The Octet Periodic Table of Natural Numbers

n	$A \equiv 8n-7$			$B \equiv 8n-6$		$C \equiv 8n-5$	$D \equiv 8n-4$				$E \equiv 8n-3$		$F \equiv 8n-2$	$G \equiv 8n-1$	$H \equiv 8n$			
	①	②	③	②	③	③	①	②	③	④	②	③	③	④	①	②	③	④
1	1			**2**		**3**	4				**5**	6		**7**		8		
2	9			10		**11**			12		**13**		14	15	16			
3		**17**		18		**19**		20				21	22	**23**			24	
4	25			26		27				28	**29**		30	**31**		32		
5			33	34		35	36				**37**		38	39		40		
6		**41**			42	**43**			44		45		46	**47**			48	
7	49			50		51		52			**53**		54	55			56	
8			57	58		**59**				60	**61**		62	63	64			
9		65			66	**67**		68				69	70	**71**		72		
10		**73**		74		75			76			77	78	**79**		80		
11	81			82		**83**			84		85		86	87			88	
12		**89**		90		91				92		93	94	95			96	
13		**97**		98		99	100				**101**		102	**103**		104		
14			105	106		**107**			108		**109**		110	111				112
15		**113**			114	115		116			117		118	119			120	
16	121			122		123				124	125		126	**127**		128		
17			129	130		**131**			132			133	134	135		136		
18		**137**			138	**139**			140			141	142	143	144			
19		145		146		147		148			**149**		150	**151**			152	
20		153			154	155				156	**157**		158	159		160		
21			161	162		**163**		164				165	166	**167**			168	
22	169			170		171			172		**173**		174	175			176	
23			177	178		**179**		180			**181**		182	183			184	
24		185			186	187				188		189	190	**191**			192	
25		**193**		194		195	196				**197**		198	**199**		200		
…																		

① Perf sq.; ② Duad sqs.; ③ Trio sqs.; ④ Quad sqs.

Tab. 2 The powers of the Operators

A	B	C	D	E	F	G	H
$A^r = A$	$B^2 = D$ $B^{r+2} = H$	$C^{2r} = C^2$ $C^{2r+1} = C$	$D^{r+1} = H$	$E^{2r} = A$ $E^{2r+1} = E$	$F^2 = D$ $F^{r+2} = H$	$G^{2r} = G^2$ $G^{2r+1} = G$	$H^r = H$

Note: $r \in \mathbf{N}$

3 Problems of the Representation of Any Positive Integer as a Sum of Squares

The problems of decomposing integers of various types into the sums of squares are one of the main topics of study in mumber theory which have not been tackled systematically as yet[1]. By introducing the new idea of periodicity of natural numbers and restating the past works of Fermat, Euler, and Lagrange, we are in a position to solve the old problem exhaustively.

3.1 Octet Discriminant

Every natural number n, $n > 1$, can be uniquely expressed in the form[2]

$$n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}$$

where $\alpha_i \in \mathbf{N}$, each p_i is a prime, and $p_i \neq p_j$ for $i \neq j$.

If we cross out pairwise the identical prime factors, the resulting expression would be

$$\boldsymbol{n} = r_1 r_2 \cdots r_s \text{ with } s \leqslant t, \text{ or } \boldsymbol{n} \equiv \varnothing$$

where r_i is some member of $\{p_1, p_2, \cdots, p_t\}$, and $\boldsymbol{n}$ is called the embryonic number of n, and $\varnothing$ the empty set.

If we replace each of the prime factors in the expression of $\boldsymbol{n}$ by the octet operator pertaining to it and group together the like operators, then we have a transient expression which can be simplified, by using Fig. 1 and Table 2, to the form

$$\Theta = q_1^{\gamma_1} \cdots q_k^{\gamma_k} \text{ or } \Theta \equiv \varnothing$$

where $\gamma_i = 1$ or 2, $k = 1, 2,$ or 3, and each q_i is an octet operator. Θ is called the discriminant because it gives the nature of the mumber of representations.

This procedure is best illustrated by the example of finding the discriminant Θ of 396805500.

Decomposing completely the given integer into prime factors, we have

$$396805500 = 3^3 \circ 13 \circ 7 \circ 19 \circ 5^3 \circ 17 \circ 2^2$$

and

$$\boldsymbol{n} = 3 \circ 13 \circ 7 \circ 19 \circ 5 \circ 17$$

Hence

$$\Theta = C \circ E \circ G \circ C \circ E \circ A = A \circ C^2 \circ E^2 \circ G = A \circ C^2 \circ A \circ G \circ C^2 G$$

The various types of the discriminant Θ are called the discriminators, amounting to 21 members, i. e., A, E, 2, $2A$, $2E$, C, C^2, G, G^2, CG, CG^2, C^2G, C^2G^2, $2C$, $2C^2$, $2G$, $2G^2$, $2CG$, $2CG^2$, $2C^2G$, $2C^2G^2$。

3.2 Behaviour of the Elements in Group C

For the integrity of our scheme, we have to restate the works of some eminent mathematicians in the simplest linguistic fashions in the sequel.

Proposition 1 No integer of the form $8n-5$ is the sum of two squares, but it can be expressed as a sum of three squares[2].

Proof Using the language of congruences, suppose that $k = 8n-5$ is a sum of two squares. Then $k \equiv 3 \pmod 8$ and $k = x^2 + y^2$ for some integers x and y.

Because integer of the form $8n-5$ is an odd number, we may let one integer, say x, be odd, then the other integer y be even.

For a positive integer $u > 1$, we have

$$u^2 \equiv \begin{cases} 1 \pmod 8, \text{for } 2 \nmid u; \\ 4 \pmod 8, \text{for } 2 \mid u, 4 \nmid u; \\ 0 \pmod 8, \text{for } 4 \mid u. \end{cases} \quad (1)$$

Thus

$$x^2 + y^2 \equiv \begin{cases} 1 \pmod 8, \text{for } 2 \nmid x, 4 \mid y, \\ 5 \pmod 8, \text{for } 2 \nmid x, 2 \mid y, 4 \nmid y. \end{cases} \quad (2)$$

But

$$8n - 5 \equiv 3 \pmod 8 \quad (3)$$

Hence

$$x^2 + y^2 \not\equiv 8 - 5 \pmod 8;$$

i. e., we cannot decompose any integer of the form $8n-5$ into a sum of two squares.

Whether an integer of the form $8n-5$ can be expressed as a sum of three squares?

Suppose that $k = 8n-5$ is the sum of three squares. Then $k \equiv 3 \pmod 8 = x^2 + y^2 + z^2$ for some integers x, y and z.

It is obvious that among the three integers, either two integers are even and one integer is odd, or all of the three are odd. Thus, from (1), we have

$$x^2 + y^2 + z^2 \equiv \begin{cases} 5 \pmod 8, \text{for } 2 \nmid x, 2 \mid y, 4 \nmid y, 4 \mid z; \\ 1 \pmod 8, \text{for } 2 \nmid x, 2 \mid y, 4 \nmid y, 2 \mid z, \\ \qquad 4 \nmid z; \textit{ or } 2 \nmid x, 4 \mid y, 4 \mid z; \\ 3 \pmod 8, \text{for } 2 \nmid x, 2 \nmid y, 2 \nmid z. \end{cases} \quad (4)$$

From the above congruences and (3), we see that

$$x^2 + y^2 + z^2 \equiv 8n - 5 \pmod 8, \quad \text{for } 2 \nmid x, 2 \nmid y \text{ and } 2 \nmid z$$

i. e., any integer of the form $8n-5$ is decomposable into a sum of three squares of odd integers.

Corollary 1 Any integer with discriminant $\Theta = C$ or CG^2 can be expressed as a sum of three squares.

3.3 Behaviour of the Elements in Group F

Proposition 1 No integer of the form $8n-2$ is the sum of two squares, but it can be expressed as a sum of three squares.

Proof Suppose that $k = 8n-2$ is the sum of two squares. Then $k \equiv 6 \pmod 8$ and $k = x^2 + y^2$ for some integers x and y.

Because integer of the form $8n-2$ is even, the integers x and y must be both even or both odd. Thus, from (1) and the relevant congruence $x^2 + y^2 \equiv 0$, 2 or $4 \pmod 8$ for the specified x and y, we see that it is impossible to make $x^2 + y^2$ congruent to $6 \pmod 8$. Hence we cannot decompose any integer of the form $8n-2$ into a sum of two squares.

Now, suppose that $k \equiv 8n-2$ can be expressed as a sum of three squares. Then $k \equiv 6 \pmod 8 = x^2 + y^2 + z^2$ fom some integers x, y, and z.

Because integer of the form $8n-2$ is even, the integers x, y and z must be either two odd and one even or all three even. Thus, from (1) and the relevant congruence $x^2+y^2+z^2\equiv 0,2,4$ *or* $6 \pmod 8$ for the specified x, y and z, we see that.

$$x^2+y^2+z^2\equiv 8n-2 \pmod 8,$$
$$\text{for } 2\nmid x, 2\nmid y, 2\mid z, 4\nmid z$$

i.e., any integer of the form $8n-2$ can be expressed as a sum of three squares, one of which is even and is not divisible by 16, and the other two are odd.

Corollary 1 Any integer with $\Theta=2C, 2G, 2C^2G$ or $2CG^2$ can be written as a sum of three squares.

3.4 Behaviour of the Elements in Group G

Proposition1 No integer of the from $8n-1$ is the sum of two or three squares, but it can be expressed as a sum of four squares[2].

Proof Since $8n-1$ is odd, we see from (2) and (4) that, the sum of two squares, say $k=x^2+y^2$, or the sum of three squares, say $k=x^2+y^2+z^2$, for some x, y and z, and $8n-1$ are incongruent modulo 8. Hence we cannot decompose an integer of the form $8n-1$ into a sum of two or three squares. But we can express it as a sum of four squares, since the sum of four squares, say $k=x^2+y^2+z^2+w^2$, is congruent to $8n-1 \pmod 8$ for $2\nmid x$, $2\nmid y$, $2\nmid z$, $2\mid w$, $4\nmid w$.

Corollary 1 Any integer with $\Theta=G$ or C^2G can be written as a sum of four squares.

3.5 Behaviours of the Elements in Groups A and E

Proposition 1 Any integer with $\Theta=A$ or E can be expressed as a sum of two squares.

Euler proved Fermat's assertion that every prime of the form $4m+1$ is decomposable uniquely into a sum of two squares[1,2]. Further, we know that if two integers are sums of two squares, then so is their product[2]. If $m>1$ is even in $4m+1$, then the prime is of the form $8n-7$ (group A); if $m\geqslant 1$ is odd, then it is of the form $8n-3$ (group E). Hence the above property is only an extension of Fermat's assertion. Note that in the general cases, the decomposition into a sum of two squares is not necessarily unique.

Proposition 2 Any integer with $\Theta=C^2, G^2$, C^2G^2, or CG can be expressed as a sum of three squares.

Judging from the distribution of representations of the sums of squares in group A (for C^2, G^2, or C^2G^2), or in group E (for CG), of the Octet Periodic table of natural numbers, this property is self-evident.

3.6 Behaviour of the Elements in Group B

The elements in group B are of the form $8n-6=2(4n-3)$. If $n=2m-1$ or $2m$ ($m=1,2,3,\cdots$), then the integers of the form $4n-3$ appear successively as elements in groups A and E, repeatedly. Consequently group B includes all the elements of groups A and E multiplied by 2.

Proposition 1 Any integer with $\Theta=2, 2A$, or $2E$ can be expressed as a sum of two squares[2].

Proposition 2 Any integer with $\Theta=2C^2, 2G^2$, $2C^2G^2$, or $2CG$ can be expressed as a sum of three squares.

This property is also self-evident by judging from the distribution of representatons of the sums of squares in group B of the Octet Periodic table of natural numbers.

3.7 Behaiour of the Elements in Group D

The elements in group D are of the form $8n-4=4(2n-1)$. If $n=4m-3, 4m-2, 4m-1$, or $4m$ ($m=1,2,3,\cdots$), then the integers of the form $2n-1$ appear successively as elements in groups A, C, E, and G, repeatedly. Consequently, group D includes all the elements of A, C, E, and G multiplied by 4. Thus, the rules of discriminant applied to groups A, C, E, and G hold equally well for group D.

3.8 Behaviour of the Elements in Group H

The elements in group H are of the form $8n=2^3n$. Since n can be any integer from 1 upward, the prime-power decomposition of n can be any combination of the primes. Thus, the rules of discriminant used for the foregoing groups hold universally for group H.

Moreover, if $n=2^{2r-1}(8t-1)$ for r, $t\in\mathbf{N}$, the integer of group H is representable as a sum of four squares. It is evident that as n increases, the additional integers which are expressible as sums of four squares become scarcer.

3.9 Summary

Summing up the foregoing results, we give below the Four Theorems of Diophantus[2]

Diophantine Theorem 1 An integer can be expressed as a sum of two squares if its discriminant Θ is A, E, 2, $2A$ or $2E$.

Diophantine Theorem 2 An integer can be expressed as a sum of four squares if its discriminant Θ is G or C^2G.

Diophantine Theorem 3 An integer can be expressed as a sum of three squares if its discriminant Θ is C, C^2, G^2, CG, CG^2, C^2G^2, $2C$, $2C^2$, $2G$, $2G^2$, $2CG$, $2CG^2$, $2C^2G$ or $2C^2G^2$. Another way of saying the same thing is that if the nonempty Θ is such that

$$\Theta \notin \{A, E, 2, 2A, 2E\} \cup \{G, C^2G\}$$

then the integer can be expressed as a sum of three squares.

Diophantine Theorem 4　For the perfect squares, the following three properties are true by intuition.

Property 1　An integer of the form $8n-7$ (group A) is a perfect square iff

$$n = (k^2 - k + 2)/2 \text{ with } k \in \mathbf{N}$$

Property 2　An integer of the form $8n-4$ (group D) is a perfect square iff

$$n = 2k^2 - 2k + 1 \text{ with } k \in \mathbf{N}$$

Property 3　An integer of the form $8n$ (group H) is a perfect square iff

$$n = 2k^2 \text{ with } k \in \mathbf{N}$$

In other words, an integer is a perfect square iff its $\Theta \equiv \varnothing$.

This gives an example that elementary mathematics can be used to solve some old problems in number theory once a highly original notion comes into our minds.

The validity of the Four Theorems of Diophantus is best illustrated by the following examples:

Example 1　Find out the representation as a sum of squares for 987654321, the ladder number of nine terraces.

Decomposing completely the given integer into prime factors, we have

$$987654321 = 3^2 \cdot 17^2 \cdot 379721$$

Thus　$\boldsymbol{n} = 379721$ and $\Theta = A$

According to Diophantine Theorem 1, this ladder number can be decomposed into a sum of two squares, i. e.

$$987654321 = 31161^2 + 4080^2$$

Example 2　Express the integer 999999996 as a sum of squares.

The prime-power decomposition of 999999996 is

$$999999996 = 2^2 \cdot 3 \cdot 83333333$$

Thus　$\boldsymbol{n} = 3 \cdot 83333333$

and　$\Theta = CE = G.$

By Diophantine Theorem 2, the given integer can be expressed as a sum of four squares, such as

$$999999996 = 2^2(15811^2 + 110^2 + 13^2 + 3^2) = 31622^2 + 220^2 + 26^2 + 6^2$$

Example 3　Express the integer 89708234 as a sum of squares.

The factorization of 89708234 into primes is

$$89708234 = 2 \cdot 7 \cdot 11 \cdot 19 \cdot 23 \cdot 31 \cdot 43 = \boldsymbol{n}$$

Thus

$$\Theta = 2 \cdot G \cdot C \cdot C \cdot G \cdot G \cdot C = 2C^3G^3 = 2CG.$$

By Diophantine Theorem 3, the given integer is expressible as a sum of three squares, such as

$$89708234 = 9469^2 + 192^2 + 97^2$$

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自然数八度周期律暨丢番图四定理

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摘　要：在发现自然数八度周期律的基础上，明确而系统地制定了丢番图四定理，从而准确、彻底、全面地解决了数论中任意正整数究竟能分解成几个平方数之和的这一古老问题。

关键词：周期表；八度周期律；八度算子；胚胎数；判别式；判符；丢番图四定理

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