

# 无穷区间上 $p$ -Laplacian 积微分方程极值解的存在性\*

方玉萍, 王颖

(临沂大学数学与统计学院, 山东 临沂 276000)

**摘要:** 带有  $p$ -Laplacian 算子的积微分方程在应用力学、天体物理和经典电学中有着广泛的应用。非线性微分方程边值问题是微分方程研究领域的一个重要分支。因此,  $p$ -Laplacian 积微分方程边值问题的研究有着巨大的理论和实际意义。系统地研究无穷区间上, 较为复杂边值条件下的一类  $p$ -Laplacian 积微分方程。利用单调迭代方法, 在适当的条件下, 不仅得到了方程极值解的存在性, 而且得到了方程解的迭代序列。最后, 通过一个例子说明了结果的实用性。

**关键词:**  $p$ -Laplacian 算子; 积微分方程; 单调迭代方法; 极值解; 无穷区间

**中图分类号:** O175.8 **文献标志码:** A **文章编号:** 0529-6579(2017)05-0041-10

## Existence of extremal solutions for the $p$ -Laplacian integro-differential equation on infinite intervals

FANG Yuping, WANG Ying

(School of Mathematics and Statistics, Linyi University, Linyi 276000, China)

**Abstract:** The integro-differential equation with  $p$ -laplacian operator is widely used in applied mechanics, astrophysics and classical electrology. The boundary value problem of nonlinear differential equations is an important branch of differential equations. Therefore, it is a great theoretical and practical significance to study the boundary value problems of  $p$ -Laplacian integro-differential equations. A class of  $p$ -Laplacian integro-differential equations with complex boundary conditions on infinite interval is systematically studied. By using the monotone iterative technique, the existence of extremal solutions as well as iterative schemes under the suitable conditions is established. At last, an example is given to demonstrate the use of the main result.

**Key words:**  $p$ -Laplacian operator; integro-differential equations; the monotone iterative technique; extremal solutions; infinite intervals

多孔介质中的湍流问题是一个基本的力学问题, 为研究此类问题 Leibenson<sup>[1]</sup> 引入了下面  $p$ -Laplacian 方程

$$(\varphi_p(x'(t)))' + f(t, x(t), x'(t)) = 0$$

其中  $\varphi_p = |x|^{p-2}x, p > 1$ 。在过去的几十年里, 很多的自然现象都可以用  $p$ -Laplacian 方程来描述, 非

牛顿力学<sup>[2]</sup>, 燃烧理论<sup>[3]</sup>, 种群生物学<sup>[4-5]</sup>, 扭转蠕变 ( $p = 2, p \rightarrow \infty$ ), 多孔介质流等<sup>[6-7]</sup>。带有 Dirichlet, Sturm Liouville 或其他非线性边值条件的  $p$ -Laplacian 方程已经被广泛研究, 同时, 由于无穷区间理论在应用数学和物理中的重要性, 许多作者开始注意到无穷区间上的  $p$ -Laplacian 方

\* 收稿日期: 2016-09-28

基金项目: 国家自然科学基金(11626125); 山东省自然科学基金(ZR2016AP04); 山东省高等学校科技计划项目(J16LI03); 临沂大学博士科研启动基金(LYDX2016BS080); 大学生创新创业项目(201610452168)

作者简介: 方玉萍(1996年生), 女; 研究方向: 非线性微分方程; E-mail: 312838088@qq.com

通信作者: 王颖(1981年生), 女; 研究方向: 非线性微分方程; E-mail: lywy1981@163.com

程<sup>[8-19]</sup>。

Kim<sup>[11]</sup>研究了下面的一维  $p$ -Laplacian 边值问题 (BVP):

$$\begin{cases} (\varphi_p(u'(t)))' + h(t)f(t, u(t)) = 0, t \in J', \\ u(0) - \sum_{i=1}^{m-2} \alpha_i u(\xi_i') = 0, \\ \lim_{t \rightarrow +\infty} u'(t) = c_\infty \geq 0 \end{cases} \quad (1)$$

其中  $J' = (0, +\infty)$ ,  $J = [0, +\infty)$ ,  $\varphi_p = |x|^{p-2}x$ ,  $p > 1$ ,  $0 \leq \alpha_i < 1$ ,  $0 < \xi'_1 < \xi'_2 < \dots < \xi'_{m-2} < +\infty$ ,  $0 \leq \sum_{i=1}^{m-2} \alpha_i < 1$ ,  $f \in C(J \times J, J)$ ,  $h \in C(J', J)$ ,  $h$  在  $t = 0$  点奇异,

$$\int_0^\infty \varphi_p^{-1} \int_s^\infty h(\tau) d\tau ds < +\infty$$

应用不动点指数理论, 文 [11] 得到 BVP (1) 正解的存在性和解的迭代序列。

文 [12] 考虑了方程

$$(\varphi_p(u'(t)))' + \varphi(t)f(t, u(t), u'(t)) = 0, \quad t \in J' \quad (2)$$

方程 (2) 的边值条件同 BVP (1), 这里  $f \in C(J \times J \times J, J)$ ,  $\varphi \in C(J, J)$ 。借助于 Avery-Peterson 不动点理论, 文 [12] 获得了方程三个正解的存在性。受以上文章的启发, 我们主要研究下面的  $p$ -Laplacian 方程:

$$\begin{cases} (\varphi_p(x'(t)))' + f(t, x(t), x'(t)), (Tx)(t), \\ (Sx)(t) = 0, \quad t \in J', \\ \alpha x(0) - \beta x'(0) - \gamma \int_0^\infty g(t)x(t)dt = a \geq 0, \\ \lim_{t \rightarrow +\infty} x'(t) = b \geq 0 \end{cases} \quad (3)$$

其中  $\varphi_p(x) = |x|^{p-2}x$ ,  $p > 1$ ,  $\varphi_q = \varphi_p^{-1}$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\alpha, \gamma > 0$ ,  $\beta \geq 0$ ,  $g \in L(J, J)$ ,  $\int_0^\infty g(t)dt < +\infty$ ,

$\int_0^\infty tg(t)dt < +\infty$ ,  $\Delta = \alpha - \gamma \int_0^\infty g(t)dt > 0$ , 且

$$\begin{aligned} (Tx)(t) &= \int_0^t K(t, s)x(s)ds, \\ (Sx)(t) &= \int_0^\infty H(t, s)x(s)ds \end{aligned} \quad (4)$$

这里  $K \in C(D, J)$ ,  $D = \{(t, s) \in J \times J; t \geq s\}$ ,  $H \in C(J \times J, J)$ ,  $f \in C(J' \times J \times J \times J \times J, J)$ ,  $f$  在  $t = 0$  点奇异。

本文主要利用单调迭代方法讨论 BVP (3) 的极值解。主要特色如下: 首先, 若  $p = 2$ ,  $g(t) = 0$ ,  $\beta = a = b = 0$ ,  $\alpha = 1$  并且  $f$  中不含有  $Tu, Su$ , 则 BVP

(3) 可以转化为下面的两点边值问题:

$$\begin{cases} x''(t) + f(t, x(t), x'(t)) = 0, t \in J', \\ x(0) = 0, \\ \lim_{t \rightarrow +\infty} x'(t) = 0, \end{cases} \quad (5)$$

文 [13] 研究了 BVP (5) 正解的存在性, 而且许多学者对于二阶微分方程做了系统的研究, 如文 [14-15]。其次, BVP (3) 中, 所研究的非线性项  $f$  是更为一般的情况, 不仅含有导数项, 而且含有  $Tu, Su$ 。最后, 研究的边值条件也更为复杂, 即两点, 三点, 多点边值问题是 BVP (3) 的特殊情况。因此, 在一定程度上推广了文 [11-13, 16-17] 的结果。

## 1 预备知识和引理

**定义 1** 若  $\alpha$  是 Banach 空间  $E$  中锥  $P$  上的连续凹函数, 则有  $\alpha: P \rightarrow [0, +\infty)$  是连续的, 并且

$$\alpha(tu + (1-t)v) \geq t\alpha(u) + (1-t)\alpha(v), \quad u, v \in P, t \in [0, 1]$$

**引理 1**<sup>[20]</sup>

$$\varphi_q(s+t) \leq \begin{cases} 2^{q-1}(\varphi_q(s) + \varphi_q(t)), q \geq 2, s, t > 0, \\ \varphi_q(s) + \varphi_q(t), 1 < q < 2, s, t > 0, \\ \varphi_q(s) > \varphi_q(t) > \varphi_q(0) = 0, q > 1, s > t > 0 \end{cases}$$

定义本文所使用的空间  $E$ :

$$E = \{x \in C^1(J, J) : \sup_{t \in J} \frac{|x(t)|}{1+t} < +\infty, \sup_{t \in J} |x'(t)| < +\infty\} \quad (6)$$

范数

$$\begin{aligned} \|x\|_D &= \max\{\|x\|_F, \|x\|_C\}, \\ \|x\|_F &= \sup_{t \in J} \frac{|x(t)|}{1+t}, \\ \|x\|_C &= \sup_{t \in J} |x'(t)| \end{aligned}$$

则  $(E, \|\cdot\|_D)$  是 Banach 空间, 定义  $E$  中的锥  $P$ :

$$P = \{x \in E; x \text{ 在 } [0, +\infty) \text{ 是凹的, 非减的, } x(t) \geq 0, x'(t) \geq 0, t \in J\}$$

本文的主要条件:

(H<sub>1</sub>)

$$\begin{aligned} \sup_{t \in J} \frac{1}{1+t} \int_0^t K(t, s)(1+s)ds &< +\infty, \\ \sup_{t \in J} \frac{1}{1+t} \int_0^\infty H(t, s)(1+s)ds &< +\infty, \end{aligned}$$

$\lim_{t' \rightarrow t} \int_0^\infty |H(t', s) - H(t, s)|(1+s)ds = 0, t, t' \in J$   
令

$$\begin{aligned} k^* &= \sup_{t \in J} \frac{1}{1+t} \int_0^t K(t, s)(1+s)ds, \\ h^* &= \sup_{t \in J} \frac{1}{1+t} \int_0^\infty H(t, s)(1+s)ds \end{aligned}$$

(H<sub>2</sub>)

$$\begin{aligned}
 f &\in C(J'J \times J \times J \times J, J), f(t, 0, 0, 0, 0) \equiv 0, t \in J, \\
 f(t, u_0, u_1, u_2, u_3) &\leq q(t)h(t, u_0, u_1, u_2, u_3), \\
 h(t, u_0, u_1, u_2, u_3) &\in C(J \times J \times J \times J, J), \\
 h(t, 0, 0, 0, 0) &\equiv 0, \\
 h(t, (1+t)u_0, u_1, (1+t)u_2, (1+t)u_3) &\text{有界,} \\
 \text{其中 } t \in J, u_i \in \bar{D} (i = 0, 1, 2), \bar{D} \subset J &\text{是闭子集,} \\
 q \in L(J, J), q(t) \equiv 0, t \in J, \\
 0 < \int_0^\infty q(t) dt < +\infty, \\
 0 < \int_0^\infty \varphi_q \left( \int_s^\infty q(\tau) d\tau \right) ds < +\infty
 \end{aligned}$$

注 1 由条件 (H<sub>1</sub>), (H<sub>2</sub>), 若  $x$  满足 BVP (3), 则

$$\begin{aligned}
 (\varphi_p(x'(t)))' &= -f(t, x(t), x'(t), \\
 (Tx)(t), (Sx)(t)) &\leq 0, t \in J
 \end{aligned}$$

即  $\varphi_p(x'(t))$  在  $J$  上非减, 也就是说,  $x'(t)$  在  $J$  上非减. 因此,  $x$  在  $J$  上是凹的.

引理 2 假设条件 (H<sub>1</sub>) 和 (H<sub>2</sub>) 成立, 则  $x \in P$  是 BVP (3) 的解当且仅当  $x \in C(J)$  是下面算子方程的不动点:

$$\begin{aligned}
 (Ax)(t) &= \frac{1}{\Delta} \left( \alpha + \beta \varphi_q \left( \int_0^\infty f(\tau, x(\tau), x'(\tau), \right. \right. \\
 (Tx)(\tau), (Sx)(\tau)) d\tau + \varphi_p(b) &+ \\
 \gamma \int_0^\infty g(t) \int_0^t \varphi_q \left( \int_s^\infty f(\tau, x(\tau), x'(\tau), (Tx)(\tau), \right. \\
 (Sx)(\tau)) d\tau + \varphi_p(b) ds dt &+ \\
 \int_0^t \varphi_q \left( \int_s^\infty f(\tau, x(\tau), x'(\tau), (Tx)(\tau), \right. \\
 (Sx)(\tau)) d\tau + \varphi_p(b) ds, t \in J &\quad (7)
 \end{aligned}$$

证明 假设  $x \in P$  是 BVP (3) 的解. 对任意的  $t \in J$ , 由 BVP (3)

$$\begin{aligned}
 \int_t^\infty (\varphi_p(x'(\tau)))' d\tau &= \\
 - \int_t^\infty f(\tau, x(\tau), x'(\tau), (Tx)(\tau), (Sx)(\tau)) d\tau &
 \end{aligned}$$

即

$$\varphi_p(x'(\infty)) - \varphi_p(x'(t)) = - \int_t^\infty \sigma(\tau) d\tau$$

由  $\lim_{t \rightarrow +\infty} x'(t) = b$  可得

$$\begin{aligned}
 x'(t) &= \varphi_q \left( \int_t^\infty f(\tau, x(\tau), x'(\tau), \right. \\
 (Tx)(\tau), (Sx)(\tau)) d\tau + \varphi_p(b) &\quad (8)
 \end{aligned}$$

将式 (8) 由 0 到  $t$  积分可得

$$\begin{aligned}
 x(t) &= x(0) + \int_0^t \varphi_q \left( \int_s^\infty f(\tau, x(\tau), x'(\tau), (Tx)(\tau), \right. \\
 (Sx)(\tau)) d\tau + \varphi_p(b) ds &
 \end{aligned}$$

由于  $\alpha x(0) - \beta x'(0) - \gamma \int_0^\infty g(t)x(t) dt = a$ , 可得到

$$\begin{aligned}
 x(t) &= \frac{1}{\Delta} \left( \alpha + \beta \varphi_q \left( \int_0^\infty f(\tau, x(\tau), x'(\tau), (Tx)(\tau), \right. \right. \\
 (Sx)(\tau)) d\tau + \varphi_p(b) &+ \\
 \gamma \int_0^\infty g(t) \int_0^t \varphi_q \left( \int_s^\infty f(\tau, x(\tau), x'(\tau), (Tx)(\tau), \right. \\
 (Sx)(\tau)) d\tau + \varphi_p(b) ds dt &+ \\
 \int_0^t \varphi_q \left( \int_s^\infty f(\tau, x(\tau), x'(\tau), (Tx)(\tau), \right. \\
 (Sx)(\tau)) d\tau + \varphi_p(b) ds, t \in J &\quad (9)
 \end{aligned}$$

对任意的  $x \in P$ , 存在  $R > 0$ , 使得  $\|x\|_D < R$ , 由条件 (H<sub>1</sub>), (H<sub>2</sub>) 可得

$$\begin{aligned}
 \int_0^\infty f(t, x(t), x'(t), (Tx)(t), (Sx)(t)) dt &\leq \\
 \int_0^\infty q(t)h(t, x(t), x'(t), (Tx)(t), (Sx)(t)) dt &\leq \\
 S_R \int_0^\infty q(t) dt < +\infty &\quad (10)
 \end{aligned}$$

其中

$$\begin{aligned}
 S_R &= \sup \{ h(t, (1+t)u_0, u_1, (1+t)u_2, (1+t)u_3) : \\
 (t, u_0, u_1, u_2, u_3) \in J \times [0, R] \times [0, R] \times & \\
 [0, k^*R] \times [0, h^*R] \} &
 \end{aligned}$$

因此, 由式 (10) 和引理 1, 有

$$\begin{aligned}
 \frac{1}{\Delta} \left( \alpha + \beta \varphi_q \left( \int_0^\infty f(\tau, x(\tau), x'(\tau), (Tx)(\tau), \right. \right. \\
 (Sx)(\tau)) d\tau + \varphi_p(b) &+ \\
 \gamma \int_0^\infty g(t) \int_0^t \varphi_q \left( \int_s^\infty f(\tau, x(\tau), x'(\tau), (Tx)(\tau), \right. \\
 (Sx)(\tau)) d\tau + \varphi_p(b) ds dt &+ \\
 \int_0^t \varphi_q \left( \int_s^\infty f(\tau, x(\tau), x'(\tau), (Tx)(\tau), \right. \\
 (Sx)(\tau)) d\tau + \varphi_p(b) ds &\leq \\
 \frac{a}{\Delta} + 2^{q-1} \left( \frac{\beta + \gamma \int_0^\infty tg(t) dt}{\Delta} + t \right) & \\
 \left. \left( \varphi_q \left( S_R \int_0^\infty q(t) dt \right) + b \right) < +\infty &\quad (11)
 \end{aligned}$$

由式 (10) 和式 (11) 可得, 式 (9) 是有定义的. 下面证明  $x$  是由式 (7) 定义的算子  $A$  的不动点.

假设  $x$  是由式 (7) 定义的算子  $A$  的不动点, 对式 (7) 直接求导数可得

$$\begin{aligned}
 x'(t) &= \varphi_q \left( \int_t^\infty f(\tau, x(\tau), x'(\tau), (Tx)(\tau), \right. \\
 (Sx)(\tau)) d\tau + \varphi_p(b) &
 \end{aligned}$$

$$(\varphi_p(x'(t)))' =$$

$$-f(t, x(t), x'(t), (Tx)(t), (Sx)(t))''$$

容易证明

$$\alpha x(0) - \beta x'(0) - \gamma \int_0^{\infty} g(t)x(t)dt = a,$$

$$\lim_{t \rightarrow +\infty} x'(t) = b$$

证明完毕。

**引理 3**<sup>[21-22]</sup> 空间  $E$  由式 (6) 定义,  $M$  是  $E$  中的有界集。若

$$\left\{ \frac{x(t)}{1+t} : x \in M \right\}, \{X'(t) : x \in M\}$$

在  $J$  上的任一有界子集上等度连续, 且对任意给定的  $\varepsilon > 0$ , 存在  $N > 0, t_1, t_2 > N$ , 使得

$$\left| \frac{x(t_1)}{1+t_1} - \frac{x(t_2)}{1+t_2} \right| < \varepsilon, |x'(t_1) - x'(t_2)| < \varepsilon$$

对  $x \in M$  一致成立, 则  $M$  在  $E$  中是相对紧的。

**引理 4** 假设条件  $(H_1), (H_2)$  成立, 则  $A:P \rightarrow P$  是全连续算子。

**证明**

(i)  $A:P \rightarrow P$  是有定义的。

通过常规的方法可以得到  $A:P \rightarrow E$  是有定义的。下面证明  $A(P) \subseteq P$ 。对任意的  $x \in P$ , 由式 (7) 可得

$$\varphi_p((Ax)'(t)) =$$

$$\int_t^{\infty} f(\tau, x(\tau), x'(\tau), (Tx)(\tau), (Sx)(\tau))d\tau + \varphi_p(b) \quad (12)$$

$$(\varphi_p((Ax)'(t)))' =$$

$$-f(t, x(t), x'(t), (Tx)(t), (Sx)(t)) \quad (13)$$

由式 (7), 式 (12), 式 (13) 和条件  $(H_2)$ , 可知  $(Ax)(t) > 0, (Ax)'(t) \geq b \geq 0, (Ax)''(t) \leq 0$ , 即  $A(P) \subseteq P$ 。

(ii)  $A:P \rightarrow P$  连续。

假设  $x_m, x \in P, \lim_{m \rightarrow +\infty} x_m = x$ , 则存在  $R_0 > 0$ , 使得  $\|x_m\|_D \leq R_0, \|x\|_D \leq R_0$ 。由于

$$\begin{aligned} & \left| \int_0^{\infty} f(\tau, x_m(\tau), x'_m(\tau), (Tx_m)(\tau), (Sx_m)(\tau))d\tau - \int_0^{\infty} f(\tau, x(\tau), x'(\tau), (Tx)(\tau), (Sx)(\tau))d\tau \right| \leq \\ & \int_0^{\infty} |f(\tau, x_m(\tau), x'_m(\tau), (Tx_m)(\tau), (Sx_m)(\tau)) + f(\tau, x(\tau), x'(\tau), (Tx)(\tau), (Sx)(\tau))|d\tau \leq \\ & \int_0^{\infty} q(\tau)(h(\tau, x_m(\tau), x'_m(\tau), (Tx_m)(\tau), (Sx_m)(\tau)) + h(\tau, x(\tau), x'(\tau), (Tx)(\tau), (Sx)(\tau)))d\tau \leq \\ & 2S_{R_0} \int_0^{\infty} q(\tau)d\tau < +\infty \end{aligned}$$

其中

$$S_{R_0} = \sup \{h(t, 1+t)u_0, u_1, (1+t)u_2, (1+t)u_3 : (t, u_0, u_1, u_2, u_3) \in$$

$J \times [0, R_0] \times [0, R_0] \times [0, k^*R_0] \times [0, h^*R_0]\}$  所以, 对任意的  $\varepsilon > 0$ , 可以找到一个充分大的  $H_0 > 0$  满足

$$S_{R_0} \left( \int_0^{\frac{1}{H_0}} q(\tau)d\tau + \int_{H_0}^{\infty} q(\tau)d\tau \right) < \frac{\varepsilon}{4}$$

由 Lebesgue 控制收敛定理和  $f$  的连续性可得

$$\begin{aligned} & \left| \int_{\frac{1}{H_0}}^{H_0} f(\tau, x_m(\tau), x'_m(\tau), (Tx_m)(\tau), (Sx_m)(\tau))d\tau - \int_{\frac{1}{H_0}}^{H_0} f(\tau, x(\tau), x'(\tau), (Tx)(\tau), (Sx)(\tau))d\tau \right| \leq \\ & \int_{\frac{1}{H_0}}^{H_0} |f(\tau, x_m(\tau), x'_m(\tau), (Tx_m)(\tau), (Sx_m)(\tau)) - f(\tau, x(\tau), x'(\tau), (Tx)(\tau), (Sx)(\tau))|d\tau \rightarrow 0, \\ & m \rightarrow +\infty \end{aligned}$$

因此对上述的  $\varepsilon > 0$ , 存在  $N > 0$ , 当  $n > N_0$  时, 有

$$\begin{aligned} & \left| \int_0^{\infty} f(\tau, x_m(\tau), x'_m(\tau), (Tx_m)(\tau), (Sx_m)(\tau))d\tau - \int_0^{\infty} f(\tau, x(\tau), x'(\tau), (Tx)(\tau), (Sx)(\tau))d\tau \right| \leq \\ & \int_0^{\infty} |f(\tau, x_m(\tau), x'_m(\tau), (Tx_m)(\tau), (Sx_m)(\tau)) - f(\tau, x(\tau), x'(\tau), (Tx)(\tau), (Sx)(\tau))|d\tau \leq \\ & \int_0^{\frac{1}{H_0}} |f(\tau, x_m(\tau), x'_m(\tau), (Tx_m)(\tau), (Sx_m)(\tau)) + f(\tau, x(\tau), x'(\tau), (Tx)(\tau), (Sx)(\tau))|d\tau + \\ & \int_{H_0}^{\infty} |f(\tau, x_m(\tau), x'_m(\tau), (Tx_m)(\tau), (Sx_m)(\tau)) + f(\tau, x(\tau), x'(\tau), (Tx)(\tau), (Sx)(\tau))|d\tau + \\ & \int_{\frac{1}{H_0}}^{H_0} |f(\tau, x_m(\tau), x'_m(\tau), (Tx_m)(\tau), (Sx_m)(\tau)) - f(\tau, x(\tau), x'(\tau), (Tx)(\tau), (Sx)(\tau))|d\tau \leq \end{aligned}$$

$$2S_{R_0} \left( \int_0^{\frac{1}{H_0}} q(\tau)d\tau + \int_{H_0}^{\infty} q(\tau)d\tau \right) +$$

$$\int_{\frac{1}{H_0}}^{H_0} |f(\tau, x_m(\tau), x'_m(\tau), (Tx_m)(\tau), (Sx_m)(\tau)) - f(\tau, x(\tau), x'(\tau), (Tx)(\tau), (Sx)(\tau))|d\tau \leq \varepsilon$$

即

$$\begin{aligned} & \left| \varphi_q \left( \int_0^{\infty} f(\tau, x_m(\tau), x'_m(\tau), (Tx_m)(\tau), (Sx_m)(\tau))d\tau + \varphi_p(b) \right) - \varphi_q \left( \int_0^{\infty} f(\tau, x(\tau), x'(\tau), (Tx)(\tau), (Sx)(\tau))d\tau + \varphi_p(b) \right) \right| \rightarrow 0, m \rightarrow +\infty \end{aligned}$$

类似地, 对任意的  $s \in J$  可得

$$\begin{aligned} & \left| \varphi_q \left( \int_s^\infty f(\tau, x_m(\tau), x'_m(\tau), (Tx_m)(\tau), \right. \right. \\ & \quad \left. \left. (Sx_m)(\tau)) d\tau + \varphi_p(b) \right) - \right. \\ & \quad \left. \varphi_q \left( \int_s^\infty f(\tau, x(\tau), x'(\tau), (Tx)(\tau), \right. \right. \\ & \quad \left. \left. (Sx)(\tau)) d\tau + \varphi_p(b) \right) \right| \rightarrow 0, m \rightarrow +\infty \end{aligned}$$

因此, 由 Lebesgue 控制收敛定理可得

$$\begin{aligned} & \| (Ax_m) - (Ax) \|_F \leq \\ & \frac{\beta}{\Delta} \left| \varphi_q \left( \int_s^\infty f(\tau, x_m(\tau), x'_m(\tau), (Tx_m)(\tau), \right. \right. \right. \\ & \quad \left. \left. (Sx_m)(\tau)) d\tau + \varphi_p(b) \right) - \right. \\ & \quad \left. \varphi_q \left( \int_s^\infty f(\tau, x(\tau), x'(\tau), (Tx)(\tau), \right. \right. \right. \\ & \quad \left. \left. (Sx)(\tau)) d\tau + \varphi_p(b) \right) \right| + \\ & \sup_{t \in J} \frac{1}{1+t} \left( \frac{\gamma}{\Delta} \int_0^\infty g(t) \int_0^t \left| \varphi_p \left( \int_s^\infty f(\tau, x_m(\tau), x'_m(\tau), \right. \right. \right. \right. \\ & \quad \left. \left. (Tx_m)(\tau), (Sx_m)(\tau)) d\tau + \varphi_p(b) \right) - \right. \\ & \quad \left. \varphi_p \left( \int_0^\infty f(\tau, x(\tau), x'(\tau), (Tx)(\tau), \right. \right. \right. \\ & \quad \left. \left. (Sx)(\tau)) d\tau + \varphi_p(b) \right) \right| ds dt + \\ & \sup_{t \in J} \frac{1}{1+t} \int_0^t \left| \varphi_q \left( \int_s^\infty f(\tau, x_m(\tau), x'_m(\tau), (Tx_m)(\tau), \right. \right. \right. \\ & \quad \left. \left. (Sx_m)(\tau)) d\tau + \varphi_p(b) \right) - \right. \\ & \quad \left. \varphi_q \left( \int_0^\infty f(\tau, x(\tau), x'(\tau), (Tx)(\tau), \right. \right. \right. \\ & \quad \left. \left. (Sx)(\tau)) d\tau + \varphi_p(b) \right) \right| ds \rightarrow 0, m \rightarrow +\infty; \\ & (Ax_m)' - (Ax)'_c = \\ & \sup_{t \in J} \left| \varphi_p \left( \int_t^\infty f(\tau, x_m(\tau), x'_m(\tau), (Tx_m)(\tau), \right. \right. \right. \\ & \quad \left. \left. (Sx_m)(\tau)) d\tau + \varphi_p(b) \right) - \right. \\ & \quad \left. \varphi_p \left( \int_t^\infty f(\tau, x(\tau), x'(\tau), (Tx)(\tau), \right. \right. \right. \\ & \quad \left. \left. (Sx)(\tau)) d\tau + \varphi_p(b) \right) \right| \rightarrow 0, m \rightarrow +\infty \end{aligned}$$

故,  $A: P \rightarrow P$  是连续的。

(iii)  $A: P \rightarrow P$  相对紧。

(a) 令  $M \subset P$  是有界子集, 则存在  $R_1 > 0$ , 使得  $\|x\|_D < R_1, x \in M$ 。所以, 对任意的  $x \in M$ , 有

$$\begin{aligned} (Ax)_F &= \sup_{t \in J} \left| \frac{(Ax)(t)}{1+t} \right| \leq \\ & \sup_{t \in J} \left| \frac{a}{\Delta} + 2^{q-1} \left( \frac{\beta + \gamma \int_0^\infty tg(t) dt}{\Delta} + t \right) \cdot \right. \\ & \quad \left. \left( \varphi_q \left( S_{R_1} \int_0^\infty q(t) dt \right) + b \right) \right| \leq \\ & \frac{a}{\Delta} + 2^{q-1} \left( \frac{\beta + \gamma \int_0^\infty tg(t) dt}{\Delta} + 1 \right) \cdot \\ & \quad \left( \varphi_q \left( S_{R_1} \int_0^\infty q(t) dt \right) + b \right) < +\infty \end{aligned} \quad (14)$$

$$\begin{aligned} (Ax)'_c &= \sup_{t \in J} | (Ax)'(t) | = \\ & \sup_{t \in J} \left| \varphi_q \left( \int_t^\infty f(\tau, x(\tau), x'(\tau), (Tx)(\tau), \right. \right. \right. \\ & \quad \left. \left. (Sx)(\tau)) d\tau + \varphi_p(b) \right) \right| \leq \\ & 2^{q-1} \left( \varphi_q \left( S_{R_1} \int_0^\infty q(t) dt \right) + b \right) < +\infty \end{aligned} \quad (15)$$

其中

$$\begin{aligned} S_{R_1} &= \sup \{ h(t, 1+t)u_0, u_1, (1+t)u_2, (1+t)u_3 : \\ & \quad (t, u_0, u_1, u_2, u_3) \in J \times [0, R_1] \times \\ & \quad [0, R_1] \times [0, h^* R_1] \times [0, h^* R_1] \} \\ & \text{由式 (14) - (15) 可得} \\ & (Ax)_D = \max \{ \| (Ax) \|_F, \| (Ax)' \|_C \} \leq \\ & \frac{a}{\Delta} + 2^{q-1} \left( \frac{\beta + \gamma \int_0^\infty tg(t) dt}{\Delta} + 1 \right) \cdot \\ & \quad \varphi_p \left( S_{R_1} \int_0^\infty q(t) dt + b \right) < +\infty \end{aligned} \quad (16)$$

由式 (16) 可知  $AM$  在  $E$  中有界。

(b) 对任意的  $T > 0, t_1, t_2 \in [0, T], x \in M$ , 不失一般性, 假设  $t_1 > t_2$ 。事实上

$$\begin{aligned} & \left| \frac{(Ax)(t_1)}{1+t_1} - \frac{(Ax)(t_2)}{1+t_2} \right| \leq \left| \frac{1}{1+t_1} - \frac{1}{1+t_2} \right| \cdot \\ & \frac{1}{\Delta} \left( \alpha + \beta \varphi_q \left( \int_0^\infty f(\tau, x(\tau), x'(\tau), (Tx)(\tau), \right. \right. \right. \\ & \quad \left. \left. (Sx)(\tau)) d\tau + \varphi_p(b) \right) + \right. \\ & \quad \left. \gamma \int_0^\infty g(t) \int_0^t \varphi_q \left( \int_s^\infty f(\tau, x(\tau), x'(\tau), (Tx)(\tau), \right. \right. \right. \\ & \quad \left. \left. (Sx)(\tau)) d\tau + \varphi_p(b) \right) ds dt \right) + \\ & \left| \frac{1}{1+t_1} \int_0^{t_1} \varphi_q \left( \int_s^\infty f(\tau, x(\tau), x'(\tau), (Tx)(\tau), \right. \right. \right. \\ & \quad \left. \left. (Sx)(\tau)) d\tau + \varphi_p(b) \right) ds - \right. \\ & \quad \left. \frac{1}{1+t_2} \int_0^{t_2} \varphi_q \left( \int_s^\infty f(\tau, x(\tau), x'(\tau), \right. \right. \right. \\ & \quad \left. \left. (Tx)(\tau), (Sx)(\tau)) d\tau + \varphi_p(b) \right) ds \right| \leq \\ & \left| \frac{1}{1+t_1} - \frac{1}{1+t_2} \right| \left( 2^{q-1} \left( \frac{\beta + \gamma \int_0^\infty tg(t) dt}{\Delta} \right) \cdot \right. \\ & \quad \left. \left( \varphi_q \left( S_{R_1} \int_0^\infty q(t) dt \right) + b \right) + \frac{a}{\Delta} \right) + \\ & \left| \frac{1}{1+t_1} - \frac{1}{1+t_2} \right| \int_0^{t_1} \varphi_q \left( S_{R_1} \int_s^\infty q(\tau) d\tau + \varphi_p(b) \right) ds + \\ & \quad \frac{1}{1+t_2} \int_{t_1}^{t_2} \varphi_q \left( S_{R_1} \int_s^\infty q(\tau) d\tau + \varphi_p(b) \right) ds \quad (17) \\ & \left| \varphi_p \left( (Ax)'(t_1) \right) - \varphi_p \left( (Ax)'(t_2) \right) \right| \leq \\ & \left| \int_{t_1}^{t_2} f(\tau, x(\tau), x'(\tau), (Tx)(\tau), (Sx)(\tau)) d\tau \right| \leq \\ & \quad S_{R_1} \left| \int_{t_1}^{t_2} q(\tau) d\tau \right| \quad (18) \end{aligned}$$

由式 (17) - (18), 对任意的  $\varepsilon > 0$ , 存在  $\delta > 0$ , 对任意的  $t_1, t_2 \in [0, b]$ ,  $|t_1 - t_2| < \delta$ , 任意的  $x \in M$ , 有

$$\begin{aligned} \left| \frac{(Ax)(t_1)}{1+t_1} - \frac{(Ax)(t_2)}{1+t_2} \right| &< \varepsilon, \\ |(Ax)'(t_1) - (Ax)'(t_2)| &< \varepsilon \end{aligned}$$

因此可得  $\left\{ \frac{(Ax)(t)}{1+t} : x \in M \right\}, \{(Ax)'(t) : x \in M\}$

在  $[0, T]$  等度连续. 由于  $T > 0$  是任意的, 所以  $\left\{ \frac{(Ax)(t)}{1+t} : x \in M \right\}, \{(Ax)'(t) : x \in M\}$  在  $J$  局部等度连续.

(c) 对任意的  $x \in M$ , 由式 (7) 可得

$$\begin{aligned} \lim_{t \rightarrow +\infty} \left| \frac{(Ax)(t)}{1+t} \right| = \\ \lim_{t \rightarrow +\infty} \frac{1}{1+t} \left( \frac{1}{\Delta} \left( \alpha + \beta \varphi_q \left( \int_0^\infty f(\tau, x(\tau), x'(\tau), \right. \right. \right. \right. \\ \left. \left. \left. (Tx)(\tau), (Sx)(\tau)) d\tau + \varphi_p(b) \right) + \right. \right. \\ \left. \left. \gamma \int_0^\infty g(t) \int_0^t \varphi_q \left( \int_s^\infty f(\tau, x(\tau), x'(\tau), \right. \right. \right. \right. \\ \left. \left. \left. (Tx)(\tau), (Sx)(\tau)) d\tau + \varphi_p(b) \right) ds dt \right) + \right. \\ \left. \int_0^t \varphi_q \left( \int_s^\infty f(\tau, x(\tau), x'(\tau), \right. \right. \right. \right. \\ \left. \left. \left. (Tx)(\tau), (Sx)(\tau)) d\tau + \varphi_p(b) \right) ds \right) \end{aligned} \quad (19)$$

由于

$$\begin{aligned} \lim_{t \rightarrow +\infty} \frac{1}{1+t} \left( \frac{1}{\Delta} \left( \alpha + \beta \varphi_q \left( \int_0^\infty f(\tau, x(\tau), x'(\tau), \right. \right. \right. \right. \\ \left. \left. \left. (Tx)(\tau), (Sx)(\tau)) d\tau + \varphi_p(b) \right) + \right. \right. \\ \left. \left. \gamma \int_0^\infty g(t) \int_0^t \varphi_q \left( \int_s^\infty f(\tau, x(\tau), x'(\tau), \right. \right. \right. \right. \\ \left. \left. \left. (Tx)(\tau), (Sx)(\tau)) d\tau + \varphi_p(b) \right) ds dt \right) \leq \\ \lim_{t \rightarrow +\infty} \frac{1}{1+t} \left( \frac{a}{\Delta} + 2^{q-1} \left( \frac{\beta + \gamma \int_0^\infty tg(t) dt}{\Delta} \right) \right. \\ \left. \left( \varphi_q \left( S_{R_1} \int_0^\infty q(t) dt \right) + b \right) \right) = 0 \end{aligned} \quad (20)$$

$$\begin{aligned} \lim_{t \rightarrow +\infty} \frac{1}{1+t} \left( \int_0^t \varphi_q \left( \int_s^\infty f(\tau, x(\tau), x'(\tau), \right. \right. \right. \right. \\ \left. \left. \left. (Tx)(\tau), (Sx)(\tau)) d\tau + \varphi_p(b) \right) ds \right) = \\ \lim_{t \rightarrow +\infty} \varphi_q \left( \int_t^\infty f(\tau, x(\tau), x'(\tau), \right. \right. \right. \right. \\ \left. \left. \left. (Tx)(\tau), (Sx)(\tau)) d\tau + \varphi_p(b) \right) = b \end{aligned} \quad (21)$$

根据式 (19) - (21) 可知

$$\lim_{t \rightarrow +\infty} \left| \frac{(Ax)(t)}{1+t} \right| = b$$

所以, 对任意的  $x \in M$ , 有

$$\left| \frac{(Ax)(t)}{1+t} - b \right| \leq$$

$$\frac{1}{1+t} \left( \frac{a}{\Delta} + 2^{q-1} \left( \frac{\beta + \gamma \int_0^\infty tg(t) dt}{\Delta} \right) \right.$$

$$\left. \left( \varphi_q \left( S_{R_1} \int_0^\infty q(t) dt \right) + b \right) \right) +$$

$$\left| \frac{1}{1+t} \left( \int_0^t \varphi_q \left( S_{R_1} \int_0^\infty q(\tau) d\tau + \varphi_p(b) \right) ds \right) - b \right| \rightarrow 0, \\ t \rightarrow +\infty \quad (22)$$

另外, 可以得到

$$\begin{aligned} \lim_{t \rightarrow +\infty} |(Ax)'(t)| = \lim_{t \rightarrow +\infty} \varphi_q \left( \int_t^\infty f(\tau, x(\tau), x'(\tau), \right. \\ \left. (Tx)(\tau), (Sx)(\tau)) d\tau + \varphi_p(b) \right) = b \end{aligned}$$

所以, 对任意的  $x \in M$ , 有

$$\begin{aligned} |(Ax)'(t) - b| = \\ \left| \varphi_q \left( \int_t^\infty f(\tau, x(\tau), x'(\tau), \right. \right. \right. \\ \left. \left. \left. (Tx)(\tau), (Sx)(\tau)) d\tau + \varphi_p(b) \right) - b \right| \leq \\ \left| \varphi_q \left( S_{R_1} \int_t^\infty q(\tau) d\tau + \varphi_p(b) \right) - b \right| \rightarrow 0, \\ t \rightarrow +\infty \end{aligned} \quad (23)$$

因此, 由式 (22) - (23), 对任意的  $\varepsilon > 0$ , 存在  $N > 0$ , 对任意的  $t > N$ , 任意的  $x \in M$ , 可得

$$\left| \frac{(Ax)(t)}{1+t} - b \right| < \frac{\varepsilon}{2}, \quad |(Ax)'(t) - b| < \frac{\varepsilon}{2}, \quad t < N$$

对任意的  $t_1, t_2 > N$ , 任意的  $x \in M$ , 有

$$\left| \frac{(Ax)(t_1)}{1+t_1} - b \right| < \frac{\varepsilon}{2}, \quad \left| \frac{(Ax)(t_2)}{1+t_2} - b \right| < \frac{\varepsilon}{2},$$

$$|(Ax)'(t_1) - b| < \frac{\varepsilon}{2}, \quad |(Ax)'(t_2) - b| < \frac{\varepsilon}{2}$$

对任意的  $t_1, t_2 > N$ , 任意的  $x \in M$ , 易知

$$\left| \frac{(Ax)(t_1)}{1+t_1} - \frac{(Ax)(t_2)}{1+t_2} \right| \leq$$

$$\left| \frac{(Ax)(t_1)}{1+t_1} - b \right| + \left| \frac{(Ax)(t_2)}{1+t_2} - b \right| < \varepsilon,$$

$$|(Ax)'(t_1) - (Ax)'(t_2)| \leq$$

$$|(Ax)'(t_1) - b| + |(Ax)'(t_2) - b| < \varepsilon$$

即  $\left\{ \frac{(Ax)(t)}{1+t} : x \in M \right\}, \{(Ax)'(t) : x \in M\}$  在  $+\infty$  一致收敛.

由引理 3, 结合  $A$  的连续性可知,  $A: P \rightarrow P$  是全连续的. 证明完毕.

## 2 主要结果

为方便起见, 令

$$m = \frac{2^{q-1}}{\Delta} \varphi_q \left( \int_0^\infty q(t) dt \right) \left( \beta + \gamma \int_0^\infty tg(t) dt + \Delta \right),$$

$$\rho = \frac{a}{\Delta} + 2^{q-1} b \left( 1 + \frac{1}{\Delta} \left( \beta + \gamma \int_0^\infty tg(t) dt \right) \right)$$

**定理 1** 假设条件  $(H_1)$ ,  $(H_2)$  成立, 且存在  $d > 2\rho$ , 有

$(H_3)$

$$\begin{aligned} f(t, u_0, u_1, u_2, u_3) &\leq f(t, \bar{u}_0, \bar{u}_1, \bar{u}_2, \bar{u}_3), t \in J, \\ 0 \leq U_0 &\leq \bar{U}_0, 0 \leq U_1 \leq \bar{U}_1, \\ 0 \leq U_2 &\leq \bar{U}_2, 0 \leq U_3 \leq \bar{U}_3 \end{aligned}$$

$(H_4)$

$$h(t, (1+t)u_0, u_1, (1+t)u_2, (1+t)u_3) \leq \varphi_p\left(\frac{d}{2m}\right),$$

$$(t, u_0, u_1, u_2, u_3) \in$$

$$J \times [0, d] \times [0, d] \times [0, k^*d] \times [0, h^*d]$$

则 BVP (3) 在  $J$  上有极大和极小解  $\mu^*, \nu^*$ , 满足

$$0 < \max\left\{\sup_{t \in J} \frac{|\mu^*(t)|}{1+t}, \sup_{t \in J} |\mu'^*(t)|\right\} \leq d,$$

$$0 < \max\left\{\sup_{t \in J} \frac{|\nu^*(t)|}{1+t}, \sup_{t \in J} |\nu'^*(t)|\right\} \leq d$$

对于

$$\mu_0(t) = d\left(\frac{t+1}{2}\right) +$$

$$2^{q-1}b\left(t + \frac{1}{\Delta}\left(\beta + \gamma \int_0^\infty tg(t)dt\right)\right) + \frac{a}{\Delta},$$

$$\nu_0(t) = 0,$$

定义迭代列  $\{\mu_n\}, \{\nu_n\}$

$$\mu_n(t) = \frac{1}{\Delta}\left(\alpha + \beta\varphi_q\left(\int_0^\infty f(\tau, \mu_{n-1}(\tau), \mu'_{n-1}(\tau),\right.\right.$$

$$\left.(T\mu_{n-1})(\tau), (S\mu_{n-1})(\tau)\right)d\tau + \varphi_p(b) +$$

$$\gamma \int_0^\infty g(t) \int_0^t \varphi_q\left(\int_s^\infty f(\tau, \mu_{n-1}(\tau), \mu'_{n-1}(\tau),\right.$$

$$\left.(T\mu_{n-1})(\tau), (S\mu_{n-1})(\tau)\right)d\tau + \varphi_p(b) dsdt +$$

$$\int_0^t \varphi_q\left(\int_s^\infty f(\tau, \mu_{n-1}(\tau), \mu'_{n-1}(\tau), (T\mu_{n-1})(\tau),\right.$$

$$\left.(S\mu_{n-1})(\tau)\right)d\tau + \varphi_p(b) ds,$$

$$\nu_n(t) = \frac{1}{\Delta}\left(\alpha + \beta\varphi_q\left(\int_0^\infty f(\tau, \nu_{n-1}(\tau), \nu'_{n-1}(\tau),\right.\right.$$

$$\left.(T\nu_{n-1})(\tau), (S\nu_{n-1})(\tau)\right)d\tau + \varphi_p(b) +$$

$$\gamma \int_0^\infty g(t) \int_0^t \varphi_q\left(\int_s^\infty f(\tau, \nu_{n-1}(\tau), \nu'_{n-1}(\tau), (T\nu_{n-1})(\tau),\right.$$

$$\left.(S\nu_{n-1})(\tau)\right)d\tau + \varphi_p(b) dsdt +$$

$$\int_0^t \varphi_q\left(\int_s^\infty f(\tau, \nu_{n-1}(\tau), \nu'_{n-1}(\tau), (T\nu_{n-1})(\tau),\right.$$

$$\left.(S\nu_{n-1})(\tau)\right)d\tau + \varphi_p(b) ds$$

并且有

$$\lim_{n \rightarrow +\infty} \max\left\{\sup_{t \in J} \frac{|\mu_n(t) - \mu^*(t)|}{1+t},\right.$$

$$\left.\sup_{t \in J} \left|\mu'_n(t) - \mu'^*(t)\right|\right\} = 0,$$

$$\lim_{n \rightarrow +\infty} \max\left\{\sup_{t \in J} \frac{|\nu_n(t) - \nu^*(t)|}{1+t},\right.$$

$$\left.\sup_{t \in J} \left|\nu'_n(t) - \nu'^*(t)\right|\right\} = 0$$

**证明** 由引理 4 知,  $A: P \rightarrow P$  是全连续的。对任意的  $x_1, x_2 \in P, x_1 \leq x_2$ , 由  $A$  的定义和条件  $(H_3)$  可知  $Ax_1 \leq Ax_2$ 。记

$$P_d = \{x \in P: \|x\|_D \leq d\}$$

下面首先证明  $A: P_d \rightarrow P_d$ 。对于  $x \in P_d$ , 有  $\|x\|_D \leq d$ , 由式 (7) 和条件  $(H_4)$  可得

$$\|(Ax)\|_F =$$

$$\sup_{t \rightarrow +\infty} \frac{1}{1+t} \left| \frac{1}{\Delta} \left( \alpha + \beta\varphi_q \left( \int_0^\infty f(\tau, x(\tau), x'(\tau),\right.\right. \right.$$

$$\left. (Tx)(\tau), (Sx)(\tau)\right) d\tau + \varphi_p(b) +$$

$$\gamma \int_0^\infty g(t) \int_0^t \varphi_q \left( \int_s^\infty f(\tau, x(\tau), x'(\tau),\right.$$

$$\left. (Tx)(\tau), (Sx)(\tau)\right) d\tau + \varphi_p(b) dsdt +$$

$$\int_0^t \varphi_q \left( \int_s^\infty f(\tau, x(\tau), x'(\tau), (Tx)(\tau),\right.$$

$$\left. (Sx)(\tau)\right) d\tau + \varphi_p(b) ds \right| \leq$$

$$\frac{a}{\Delta} + \frac{1}{\Delta} \left( 2^{q-1} \varphi_q \left( \int_0^\infty q(t) dt \right) \cdot \right.$$

$$\left. \left( \beta + \gamma \int_0^\infty tg(t) dt + \Delta \right) \right) \frac{d}{2m} +$$

$$2^{q-1}b \left( 1 + \frac{1}{\Delta} \left( \beta + \gamma \int_0^\infty tg(t) dt \right) \right) \leq d,$$

$$\|(Ax)'\|_D = \sup_{t \in J} |(Ax)'(t)| =$$

$$\sup_{t \in J} \left| \varphi_q \left( \int_t^\infty f(\tau, x(\tau), x'(\tau), (Tx)(\tau),\right.\right.$$

$$\left. (Sx)(\tau)\right) d\tau + \varphi_p(b) \right| \leq$$

$$2^{q-1} \left( \varphi_q \left( \int_0^\infty q(t) dt \right) \frac{d}{2m} + b \right) \leq d$$

因此有  $(Ax)_D = \max\{\|(Ax)\|_F, \|(Ax)'\|_D\} \leq d$ , 即  $A: P_d \rightarrow P_d$ 。由于

$$u_0(t) = d\left(\frac{t+1}{2}\right) +$$

$$2^{q-1}b\left(t + \frac{1}{\Delta}\left(\beta + \gamma \int_0^\infty tg(t)dt\right)\right) + \frac{a}{\Delta}, t \in J$$

则有  $\mu_0 \in P_d$ 。令

$$\mu_1 = A\mu_0, \mu_2 = A\mu_1 = A^2\mu_0$$

由引理 4 可知  $\mu_1 \in P_d, \mu_2 \in P_d$ 。定义

$$\mu_{n+1} = A\mu_n = A^{n+1}\mu_0, n = 0, 1, 2, \dots$$

由于  $A: P_d \rightarrow P_d$ , 则有  $\mu_n \in A(P_d) \subset P_d, n = 0, 1, 2, \dots$ 。由  $A$  的全连续性可得  $\{\mu_n\}_{n=1}^\infty$  是列紧集, 因此  $\{\mu_n\}_{n=1}^\infty$  有收敛子列  $\{\mu_{n_k}\}_{n=1}^\infty$  且存在  $\mu^* \in P_d$ , 使得  $\mu_{n_k} \rightarrow \mu^*$ 。由式 (7) 和条件  $(H_4)$  可得

$$\mu_1(t) = (A\mu_0)(t) = \frac{1}{\Delta} \left( \alpha + \beta\varphi_q \left( \int_0^\infty f(\tau, \mu_0(\tau),\right.\right.$$

$$\begin{aligned} & \mu_0'(\tau), (T\mu_0)(\tau), (S\mu_0)(\tau) d\tau + \varphi_p(b) + \\ & \gamma \int_0^\infty g(t) \int_0^t \varphi_q \left( \int_s^\infty f(\tau, \mu_0(\tau), \mu_0'(\tau), (T\mu_0)(\tau), \right. \\ & \quad \left. (S\mu_0)(\tau)) d\tau + \varphi_p(b) \right) ds dt + \\ & \int_0^t \varphi_q \left( \int_s^\infty f(\tau, \mu_0(\tau), \mu_0'(\tau), (T\mu_0)(\tau), \right. \\ & \quad \left. (S\mu_0)(\tau)) d\tau + \varphi_p(b) \right) ds \leq \\ & \frac{a}{\Delta} + \frac{1}{\Delta} \left( 2^{q-1} \left( \varphi_q \left( \int_0^\infty q(t) dt \right) \frac{d}{2m} + b \right) \cdot \right. \\ & \quad \left. \left( \beta + \gamma \int_0^\infty tg(t) dt \right) \right) + \\ & 2^{q-1} \left( b + \varphi_q \left( \int_0^\infty q(t) dt \right) \frac{d}{2m} \right) t \leq d \left( \frac{t+1}{2} \right) + \\ & 2^{q-1} b \left( t + \frac{1}{\Delta} \left( \beta + \gamma \int_0^\infty tg(t) dt \right) \right) + \frac{a}{\Delta} = \mu_0(t), \\ & \quad t \in J \end{aligned} \quad (24)$$

$$\begin{aligned} \mu_1'(t) &= (A\mu_0)'(t) = \varphi_q \left( \int_t^\infty f(\tau, \mu_0(\tau), \mu_0'(\tau), \right. \\ & \quad \left. (T\mu_0)(\tau), (S\mu_0)(\tau)) d\tau + \varphi_p(b) \right) \leq \\ & 2^{q-1} \left( \varphi_q \left( \int_0^\infty q(t) dt \right) \frac{d}{2m} + b \right) \leq \\ & \frac{d}{2} + 2^{q-1} b = \mu_0'(t), \quad t \in J \end{aligned} \quad (25)$$

因此, 有式 (24) - (25) 和条件 (H<sub>4</sub>) 可知

$$\begin{aligned} \mu_2(t) &= (A\mu_1)(t) \leq (A\mu_0)(t) = \mu_1(t), \quad t \in J; \\ \mu_2'(t) &= (A\mu_1)'(t) \leq (A\mu_0)'(t) = \mu_1'(t), \quad t \in J \end{aligned} \quad (26)$$

故

$$\begin{aligned} \mu_{n+1}(t) &\leq \mu_n(t), \mu_{n+1}'(t) \leq \mu_n'(t), \\ & \quad t \in J, n = 0, 1, 2, \dots \end{aligned} \quad (27)$$

所以, 存在  $\mu^* \in P_d$ , 使得  $\mu_n \rightarrow \mu^*, n \rightarrow +\infty$ 。应用  $A$  的连续性和  $\mu_{n+1} = A\mu_n$  可知  $A\mu^* = \mu^*$ 。令  $\nu_0(t) = 0, t \in J$ , 则  $\nu_0(t) \in P_d$ 。令  $\nu_1 = A\nu_0, \nu_2 = A^2\nu_0$ , 由引理 4 可得  $\nu_1 \in P_d, \nu_2 \in P_d$ 。定义  $\nu_{n+1} = A\nu_n = A^{n+1}\nu_0, n = 0, 1, 2, \dots$ 。由于  $A: P_d \rightarrow P_d$ , 因此有  $\nu_n \in A(P_d) \subset P_d, n = 0, 1, 2, \dots$ 。根据  $A$  的全连续性,  $\{\nu_n\}_{n=1}^\infty$  是列紧集, 故  $\{\nu_{n_k}\}_{n_k=1}^\infty$  有收敛子列  $\{\nu_{n_k}\}_{n_k=1}^\infty$ , 且存在  $\mu^* \in P_d$ , 使得  $\nu_{n_k} \rightarrow \mu^*$ 。由  $\nu_1 = A\nu_0 \in P_d$  可知

$$\begin{aligned} \nu_1(t) &= (A\nu_0)(t) = (A0)(t) \geq 0 = \nu_0(t), \quad t \in J; \\ \nu_1'(t) &= (A\nu_0)'(t) = (A0)'(t) \geq 0 = \nu_0'(t), \\ & \quad t \in J \end{aligned}$$

根据条件 (H<sub>3</sub>), (H<sub>4</sub>) 可得

$$\begin{aligned} \nu_2(t) &= (A\nu_1)(t) = (A0)(t) \geq 0 = \nu_1(t), \quad t \in J; \\ \nu_2'(t) &= (A\nu_1)'(t) = (A0)'(t) \geq 0 = \nu_1'(t), \\ & \quad t \in J \end{aligned}$$

因此有

$$\begin{aligned} \nu_{n+1}(t) &\geq \nu_n(t), \nu_{n+1}'(t) \geq \nu_n'(t), \\ & \quad t \in J, n = 0, 1, 2, \dots \end{aligned} \quad (28)$$

所以存在  $\nu^* \in P_d$ , 满足  $\nu_n \rightarrow \nu^*, n \rightarrow +\infty$ 。应用  $A$  的连续性和  $\nu_{n+1} = A\nu_n$  可知  $A\nu^* = \nu^*$ 。

下面证明  $\mu^*, \nu^*$  是 BVP (3) 在区间  $\left(0, d\left(\frac{t+1}{2}\right) + 2^{q-1}b\left(t + \frac{1}{\Delta}\left(\beta + \gamma \int_0^\infty tg(t) dt\right)\right) + \frac{a}{\Delta}\right)$  上的极大和极小解。

令

$$\begin{aligned} u &\in \left(0, d\left(\frac{t+1}{2}\right) + \right. \\ & \quad \left. 2^{q-1}b\left(t + \frac{1}{\Delta}\left(\beta + \gamma \int_0^\infty tg(t) dt\right)\right) + \frac{a}{\Delta}\right) \end{aligned}$$

是 BVP (3) 的任一解, 则有  $Au = u$ 。由于  $A$  是非减的, 并且

$$\nu_0(t) = 0 \leq u(t) \leq d\left(\frac{t+1}{2}\right) +$$

$$2^{q-1}b\left(t + \frac{1}{\Delta}\left(\beta + \gamma \int_0^\infty tg(t) dt\right)\right) + \frac{a}{\Delta} = \mu_0(t)$$

从而有

$$\nu_1(t) = (A\nu_0)(t) \leq u(t) \leq (A\mu_0)(t) = \mu_1(t), \quad t \in J$$

因此可得

$$\nu_n(t) \leq \mu(t) \leq \mu_n(t), n = 1, 2, 3, \dots \quad (29)$$

由于  $\mu^* = \lim_{n \rightarrow +\infty} \mu_n, \nu^* = \lim_{n \rightarrow +\infty} \nu_n$ , 由式 (24) - (29) 可知

$$\begin{aligned} \nu_0 &\leq \nu_1 \leq \dots \leq \nu_n \leq \dots \leq \nu^* \leq u \leq \\ & \quad \mu^* \leq \dots \leq \mu_n \leq \dots \leq \mu_1 \leq \mu_0 \end{aligned} \quad (30)$$

由于  $f(t, 0, 0, 0, 0) \equiv 0, t \in J$ , 所以零不是 BVP (3) 的解。故由式 (30) 可知结论成立, 证明完毕。

**注 2** 定理 1 中的迭代列分别由一个简单的函数和零函数开始的, 这在实际应用中是十分方便的。

类似地可以得到下面的定理 2。

**定理 2** 假设条件 (H<sub>1</sub>) - (H<sub>3</sub>) 成立, 并且存在  $d_n > d_{n-1} > \dots > d_2 > d_1 > 2\rho$  满足下面的条件:

$$(H_4)$$

$$h(t, (1+t)u_0, u_1, (1+t)u_2, (1+t)u_3) \leq \varphi_p\left(\frac{d_k}{2m}\right),$$

$$\begin{aligned} (t, u_0, u_1, u_2, u_3) &\in J \times [0, d_k] \times [0, d_k] \times \\ & \quad [0, k^* d_k] \times [0, k^* d_k], \end{aligned}$$

$$k = 1, 2, \dots, n$$

则 BVP (3) 有极大和极小解  $\mu_k^*, \nu_k^*$ , 且

$$0 < \max \left\{ \sup_{t \in J} \frac{|\mu_k^*(t)|}{1+t}, \sup_{t \in J} |\mu_k^*(t)| \right\} \leq d,$$

$$0 < \max \left\{ \sup_{t \in J} \frac{|\nu_k^*(t)|}{1+t}, \sup_{t \in J} |\nu_k'^*(t)| \right\} \leq d$$

对于

$$\mu_{k0}(t) = d_k \left( \frac{t+1}{2} \right) +$$

$$2^{q-1} b \left( t + \frac{1}{\Delta} \left( \beta + \gamma \int_0^\infty t g(t) dt \right) \right) = \frac{a}{\Delta}, \nu_{k0}(t) = 0$$

定义迭代列  $\{\mu_{kn}\}, \{\nu_{kn}\}$

$$\begin{aligned} \mu_{kn}(t) &= \frac{1}{\Delta} \left( \alpha + \beta \varphi_q \left( \int_0^\infty f(\tau, \mu_{k(n-1)}(\tau), \mu'_{k(n-1)}(\tau), \right. \right. \\ &\quad \left. \left. (T\mu_{k(n-1)})(\tau), (S\mu_{k(n-1)})(\tau)) d\tau + \varphi_p(b) \right) + \right. \\ &\quad \left. \gamma \int_0^\infty g(t) \int_0^t \varphi_q \left( \int_s^\infty f(\tau, \mu_{k(n-1)}(\tau), \mu'_{k(n-1)}(\tau), \right. \right. \right. \\ &\quad \left. \left. (T\mu_{k(n-1)})(\tau), (S\mu_{k(n-1)})(\tau)) d\tau + \varphi_p(b) \right) ds dt \right) + \\ &\quad \int_0^t \varphi_q \left( \int_s^\infty f(\tau, \mu_{k(n-1)}(\tau), \mu'_{k(n-1)}(\tau), (T\nu_{k(n-1)})(\tau), \right. \\ &\quad \left. (S\nu_{k(n-1)})(\tau)) d\tau + \varphi_p(b) \right) ds, \\ \nu_{kn}(t) &= \frac{1}{\Delta} \left( \alpha + \beta \varphi_q \left( \int_0^\infty f(\tau, \nu_{k(n-1)}(\tau), \nu'_{k(n-1)}(\tau), \right. \right. \\ &\quad \left. \left. (T\nu_{k(n-1)})(\tau), (S\nu_{k(n-1)})(\tau)) d\tau + \varphi_p(b) \right) + \right. \\ &\quad \left. \gamma \int_0^\infty g(t) \int_0^t \varphi_q \left( \int_s^\infty f(\tau, \nu_{k(n-1)}(\tau), \nu'_{k(n-1)}(\tau), \right. \right. \right. \\ &\quad \left. \left. (T\nu_{k(n-1)})(\tau), (S\nu_{k(n-1)})(\tau)) d\tau + \varphi_p(b) \right) ds dt \right) + \\ &\quad \int_0^t \varphi_q \left( \int_s^\infty f(\tau, \nu_{k(n-1)}(\tau), \nu'_{k(n-1)}(\tau), \right. \\ &\quad \left. (T\nu_{k(n-1)})(\tau), (S\nu_{k(n-1)})(\tau)) d\tau + \varphi_p(b) \right) ds \end{aligned}$$

且

$$\begin{aligned} \lim_{n \rightarrow +\infty} \max \left\{ \sup_{t \in J} \frac{|\mu_{kn}(t) - \mu_k^*(t)|}{1+t}, \right. \\ \left. \sup_{t \in J} \left| \mu'_{kn}(t) - \mu_k'^*(t) \right| \right\} = 0, \\ \lim_{n \rightarrow +\infty} \max \left\{ \sup_{t \in J} \frac{|\nu_{kn}(t) - \nu_k^*(t)|}{1+t}, \right. \\ \left. \sup_{t \in J} \left| \nu'_{kn}(t) - \nu_k'^*(t) \right| \right\} = 0 \end{aligned}$$

注 3 定理 1 中的解  $\mu^*, \nu^*$  可能重合, 这时 BVP (3) 在  $P_d$  上只有一个解。类似地定理 2 中的解也可能重合。

### 3 例子

考虑 BVP

$$\begin{aligned} &(|x'(t)| |x'(t)|)' + \\ &e^{-6t} \left( 10^{-2} \left( \frac{x(t)}{1+t} \right)^2 + \frac{x'(t)}{50} + \right. \\ &\left. \frac{10^{-1}}{(1+t)^2} \left( \int_0^t \frac{x(s)}{(1+t+s)^2(1+s)} ds + \right. \right. \\ &\left. \left. \int_0^\infty \frac{\cos^2(t-s)x(s)}{(1+s^2)(1+s)} ds \right) \right) = 0, t \in J'; \end{aligned}$$

$$\begin{aligned} 89x(0) - 3x'(0) - 2 \int_0^\infty \frac{1}{(1+t)^3} x(t) dt = 2; \\ \lim_{t \rightarrow +\infty} x'(t) = 1 \end{aligned}$$

显然

$$p = 3, q = \frac{3}{2}, \alpha = 89, \beta = 3, \gamma = 2, a = 2, b = 1,$$

$$g(t) = \frac{1}{(1+t)^3}, \int_0^\infty g(t) dt = \frac{1}{2},$$

$$\Delta = \alpha - \gamma \int_0^\infty g(t) dt = 88,$$

$$K(t, s) = \frac{1}{(1+t+s)^2(1+s)},$$

$$H(t, s) = \frac{\cos^2(t-s)}{(1+s^2)(1+s)}$$

通过计算, 可得

$$\begin{aligned} \int_0^\infty |H(t', s) - H(t, s)| (1+s) ds &\leq \\ \int_0^\infty \frac{|\cos^2(t'-s) - \cos^2(t-s)|}{(1+s^2)} ds &= \\ \frac{\pi}{2} |t' - t| \rightarrow 0, t' \rightarrow t; \\ k^* &= \sup_{t \in J} \frac{1}{1+t} \int_0^t \frac{1+s}{(1+t+s)^2(1+s)} ds = \\ \sup_{t \in J} \left( \frac{1}{(1+t)^2} - \frac{1}{(1+t)(1+2t)} \right) &\leq 1, \\ h^* &= \sup_{t \in J} \frac{1}{1+t} \int_0^\infty \frac{\cos^2(t-s)(1+s)}{(1+s^2)(1+s)} ds \leq \\ \sup_{t \in J} \frac{1}{1+t} \int_0^\infty \frac{1}{1+s^2} ds &= \frac{\pi}{2} \end{aligned}$$

故条件  $(H_1)$  成立。由

$$\begin{aligned} f(t, u_0, u_1, u_2, u_3) &= \\ e^{-6t} \left( 10^{-2} \left( \frac{u_0}{1+t} \right)^2 + \frac{u_1}{50} + \right. \\ \left. \frac{10^{-1}}{(1+t)^2} \left( \int_0^t \frac{u_2}{(1+t+s)^2(1+s)} ds + \right. \right. \\ \left. \left. \int_0^\infty \frac{\cos^2(t-s)u_3}{(1+s^2)(1+s)} ds \right) \right), \end{aligned}$$

可知  $q(t) = e^{-6t}$ ,

$$\begin{aligned} h(t, u_0, u_1, u_2, u_3) &= 10^{-2} \left( \frac{u_0}{1+t} \right)^2 + \frac{u_1}{50} + \\ \frac{10^{-1}}{(1+t)^2} \left( \int_0^t \frac{u_2}{(1+t+s)^2(1+s)} ds + \right. \\ \left. \int_0^\infty \frac{\cos^2(t-s)u_3}{(1+s^2)(1+s)} ds \right) \end{aligned}$$

因此有

$$\int_0^\infty q(t) dt = \frac{1}{6}, \int_0^\infty \varphi_q \left( \int_s^\infty q(\tau) d\tau \right) ds = \frac{\sqrt{3}}{18}$$

即条件  $(H_2)$  成立。由于

$$\int_0^{\infty} tg(t) dt = \frac{1}{2}, \varphi_q\left(\int_0^{\infty} q(\tau) d\tau\right) = \frac{\sqrt{6}}{6}$$

可得  $m = \frac{23\sqrt{3}}{66}, \rho = \frac{1}{22}\left(\frac{1}{2} + 23\sqrt{2}\right)$ 。取  $d = \sqrt{3}$ , 有

$$\varphi_p\left(\frac{d}{2m}\right) = \varphi_p\left(\frac{33}{23}\right) = \frac{1089}{529} \approx 2.06$$

且对于

$$(t, u_0, u_1, u_2, u_3) \in$$

$$J \times [0, d] \times [0, d] \times [0, k^*d] \times [0, h^*d],$$

$h$  满足

$$h(t, (1+t)u_0, (1+t)u_1, (1+t)u_2, (1+t)u_3) \leq$$

$$10^{-2} \cdot 3 + \sqrt{3}\left(\frac{1}{50} + 10^{-1}\left(1 + \frac{\pi}{2}\right)\right) \approx 0.51 < \varphi_p\left(\frac{d}{2m}\right)$$

因此, 定理 1 的所有条件都满足。故定理 1 的结论成立。

### 参考文献:

- [1] LEIBENSON L S. General problem of the movement of a compressible fluid in a porous medium[J]. *Izvestiia Akademii Nauk Kirgizskoi SSSR*, 1983, 9: 7-10.
- [2] DIAZ J I, THELIN F D. On a nonlinear parabolic problem arising in some models related to turbulent flows [J]. *SIAM Journal on Mathematical Analysis*, 1994, 25(4): 1085-1111.
- [3] RAMASWAMY M, SHIVAJI R. Multiple positive solutions for classes of  $p$ -Laplacian equations [J]. *Differential Integral Equations*, 2004, 17: 1255-1261.
- [4] ORUGANTI S, SHI J, SHIVAJI R. Diffusive logistic equation with constant yield harvesting I: Steady-states [J]. *Transactions of the American Mathematical Society*, 2002, 354(9): 3601-3619.
- [5] ORUGANTI S, SHI J, SHIVAJI R. Logistic equation with  $p$ -Laplacian and constant yield harvesting [J]. *Abstract and Applied Analysis*, 2004, 2004(9): 723-727.
- [6] GLOWINSKI R, RAPPAZ J. Approximation of a nonlinear elliptic problem arising in a non-Newtonian fluid flow model in glaciology [J]. *Esaim Mathematical Modelling Numerical Analysis*, 2003, 37: 175-186.
- [7] CIRSTEA F, MOTREANU D, RADULESCU V. Weak solutions of quasilinear problems with nonlinear boundary condition [J]. *Nonlinear Analysis*, 2001, 43(5): 623-636.
- [8] ZHAO X, GE W. Existence of at least three positive solutions for multi-point boundary value problem on infinite intervals with  $p$ -Laplacian operator [J]. *Journal of Applied Mathematics Computing*, 2008, 28: 391-403.
- [9] SUN J, CHEN H. Multiple positive solutions for multi-point boundary value problems with a  $p$ -Laplacian on the half-line [J]. *Journal of Applied Mathematics Computing*, 2010, 33: 173-191.
- [10] LIANG S, ZHANG J. The existence of countably many

- positive solutions for one-dimensional  $p$ -Laplacian with infinitely many singularities on the half-line [J]. *Applied Mathematics and Computation*, 2008, 201(1): 210-220.
- [11] KIM C G. Existence and iteration of positive solutions for multi-point boundary value problems on a half-line [J]. *Computers and Mathematics with Applications*, 2011, 61(7): 1898-1905.
  - [12] GUO Y, YU C, WANG J. Existence of three positive solutions for  $m$ -point boundary value problems on infinite intervals [J]. *Nonlinear Analysis*, 2009, 71: 717-722.
  - [13] LIU Y. Existence and unboundedness of positive solutions for singular boundary value problems on half-line [J]. *Applied Mathematics and Computation* 2003, 144(2): 543-556.
  - [14] LIU B, LI J, LIU L. Existence and uniqueness for an  $m$ -point boundary value problem at resonance on infinite intervals [J]. *Computers and Mathematics with Applications*, 2012, 64(6): 1677-1690.
  - [15] LIAN H, GENG F. Multiple unbounded solutions for a boundary value problem on infinite intervals [J]. *Boundary Value Problems*, 2011, 2011(1): 1-8.
  - [16] LIU Y, WONG P. Unbounded solutions of BVP for second order ODE with  $p$ -laplacian on the half line [J]. *Applications of Mathematics*, 2013, 58(2): 179-204.
  - [17] LI J, LIU B, LIU L. Solutions for a boundary value problem at resonance on  $[0, +\infty)$  [J]. *Mathematical Computer Modelling*, 2013, 58: 1769-1776.
  - [18] 程伟, 徐家发. 一类分数阶哈密顿系统非平凡解的存在性[J]. *中山大学学报(自然科学版)*, 2016, 55(5): 21-26.  
CHEN W, XU J F. Existence of nontrivial solutions for a class of fractional Hamiltonian systems [J]. *Acta Scientiarum Naturalium Universitatis Sunyatseni*, 2016, 55(5): 21-26.
  - [19] 王金华, 向红军, 赵育林. 一类分数阶差分方程边值问题解的存在性及 Ulam 稳定性[J]. *中山大学学报(自然科学版)*, 2016, 55(2): 1-6.  
WANG J H, XIANG H J, ZHAO Y L. Existence and Ulam stability of solutions for a boundary value problem of nonlinear fractional difference equation [J]. *Acta Scientiarum Naturalium Universitatis Sunyatseni*, 2016, 55(2): 1-6.
  - [20] JIANG J, LIU L, WU Y. Positive solutions for  $p$ -Laplacian fourth-order differential system with integral boundary conditions [J]. *Discrete Dynamics in Nature and Society*, 2012, 2012(11): 715-735.
  - [21] LIU Y. Boundary value problem for second order differential equations on unbounded domain [J]. *Acta Analysis Functionalis Applicata*, 2002, 4(3): 211-216.
  - [22] CORDUNEANU C. *Integral equations and stability of feedback systems* [M]. New York: Academic Press, 1973.