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五维空间 Navier-Stokes 方程的正则性*

马西霞

（中国工程物理研究院研究生院，北京 100088）

摘　要： 利用紧性定理研究五维空间 Navier-stokes 方程的正则性。证明了如果 $u \in L^{4,\infty}(\Omega \times (-T_1,0))$ 是一个 Leray-Hopf 弱解，并且 $\int_{\Omega\times(-T_1,0)} u^3 + p^{\frac{3}{2}} < \varepsilon$，那么 u 是 Hölder 连续。

关键词： 五维空间；Navier-stokes 方程；紧性定理；Hölder 连续

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The regularity of Navier-Stokes equations in five-dimensional space

MA Xixia

(Graduate School of China Academy of Engineering Physics, Beijing 100088, China)

Abstract: Using the compactness theorem, the regularity of Navier-Stokes equations in five-dimensional space is studied. It is proven that if $u \in L^{4,\infty}(\Omega \times (-T_1,0))$ is a Leray-Hopf weak solution and $\int_{\Omega\times(-T_1,0)} u^3 + p^{\frac{3}{2}} < \varepsilon$, then u is Hölder continuous.

Key words: five dimensional space; Navier-Stokes; compact theorem; Hölder continuous

This paper is concerned with the partial regularity of weak solutions of incompressible Navier-Stokes equations in five dimensional space with unit viscosity and zero external force:

$$\begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0, \\ \nabla \cdot u = 0 \end{cases} \tag{1}$$

for $x \in \Omega \subseteq R^5, t < 0$, and

$$\begin{cases} u(x,0) = u_0(x), x \in \Omega; \\ u(x,t) = 0, x \in \partial\Omega, -T_1 < t < 0 \end{cases} \tag{2}$$

The concepts of weak solutions of (1)-(2), and their regularity were already introduced in the fundamental paper of J. Leray. Pioneering works of J. Leray showed the existence of a function u and p such that

(**i**) $u \in L^{2,\infty}(Q) \cap L^2(-T_1,0;H^1(B))$;

(**ii**) the function $t \to \int_B uw\mathrm{d}x$ is continuous in $[-T_1,0]$, for $w \in L^2$.

(**iii**) u satisfies the Navier-Stokes equations in the distribution sense.

There are many important results concerning the regularity of weak solutions. Among them, we wish to mention the works of Serrin, which asserts that if $u \in L^q((-T_1,0)L^p(\Omega))$ satisfies $\frac{5}{p} + \frac{2}{q} < 1$, then $u$is smooth in the spatial direction. This result was later improved by the reference [1-3] to the inequality with the critical points.

In the series of papers [1-2, 4-5], when the spatial dimension d is 3, Scheffer introduced the notions of suitable weak solutions and the generalized energy inequality. He also established various partial regularity results of such weak solutions. Scheffer's results

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作者简介： 马西霞（1990 年生），女；**研究方向：** 流体方程；E-mail：kfmaxixia@163.com

were further generalized and strengthened in the paper of Caffareli, Kohn and Nirenberg[2], for $d = 3$.

For $d = 4$, V. Scheffer[6] proved that there exists a weak solution u in $\mathbf{R}^4 \times \mathbf{R}$ such that u is continuous outside a locally closed set of $\mathbf{R}^4 \times \mathbf{R}$ whose 3-D Hausdorff measure is finite. For $d = 5,6$, Struwe[2], Du and Dong[3] obtained the corresponding results in the steady Navier-Stokes equations. Tian and Xin[7] showed the partial regularity for smooth solutions and any spatial dimension in the steady Navier-Stokes equations.

Now let's state our results. Similarly dealing with weak solutions, our result is about the partial regularity of the weak solutions in five-dimensional space. We show this after two compactness theorems are obtained. Our proof is conceptually similar to Lin's in [4], but the problem is that we don't use the Sobolev embedding theorem. The main difficulty is that how to obtain the compactness theorem. We overcome it by the interpolation theory in harmonic analysis and the condition $u \in L^{4,\infty}(\Omega \times (-T_1, 0))$. It is possible because the strong lower-level convergence and the weak higher-level convergence imply the strong convergence between them.

1 The Compactness theorem

As we all know, Fanghua Lin's or Caffarelli-kohn-Nirenberg's method is not directly applied in five spatial dimension, because the Sobolev embedding theorem can not be used and it is not clear that the iteration procedure is valid. Here we give a compactness theorem through the interpolation theory and the condition $u \in L^{4,\infty}(\Omega \times (-T_1, 0))$.

Definition 1 Let Ω be a open set in $\mathbf{R}^5$. We say that a pair (u,p) is a suitable weak solution to the Navier-Stokes equations on the set $\Omega \times (-T_1, 0)$ if it satisfies the conditions:

(i)

$$u \in L^{2,\infty}(\Omega \times (-T_1,0)) \cap L^2(-T_1,0;H^1(\Omega)),$$

$$p \in L^{\frac{3}{2}}(\Omega \times (-T_1,0)) \tag{3}$$

(ii) u and p satisfy the Navier-Stokes equations in the distribution sense;

(iii) u and p satisfy the local energy inequality

$$\int_\Omega \varphi |u|^2 + 2\int_{\Omega\times(-T_1,t)} \varphi |\nabla u|^2 \mathrm{d}x\mathrm{d}t' \leqslant \int_{\Omega\times(-T_1),t} (|u|^2(\Delta\varphi + \partial_t\varphi) + u\cdot\nabla\varphi(|u|^2 + 2p))\mathrm{d}x\mathrm{d}t' \tag{4}$$

for a. a. $t \in (-T_1, 0)$ and for all nonnegative functions $\varphi \in C_0^\infty(\Omega \times (-T_1, 0))$.

Theorem 1 [5] Let X_0, X and X_1 be three Banach spaces and $X_i (i = 0,1)$ is reflective such that

$$X_0 \subseteq X \subseteq X_1$$

the injection of X into X_1 being continuous; and the injection of X_0 into X is compact. Let T be a fixed number, and let α_0, α_1 be two finite numbers such that $\alpha_i \geqslant 1, i = 0,1$.

We consider the space

$$Y = \left\{ v \in L^{\alpha_0}(-T,0;X_0), \; v' = \frac{\mathrm{d}v}{\mathrm{d}t} \in L^{\alpha_1}(-T,0;X_1) \right\}$$

And the space Y is provided with the norm

$$\|v\|_Y = \|v\|_{L^{\alpha_0}(-T,0;X_0)} + \|v\|_{L^{\alpha_1}(-T,0;X_1)}$$

Then the injection of Y into $L^\alpha(0, T; X)$ is compact.

Lemma 1 Let (u,p) is a weak solution of the Cauchy problems of the Navier-Stokes equations in Ω with $u \in L^{2,\infty}(\Omega \times (-T_1,0)) \cap L^2(-T_1,0; H^1(\Omega))$. In addition,

$$u \in L^{4,\infty}(\Omega \times (-T_1,0)) \tag{5}$$

then $p \in L^{\frac{35}{23}}(\Omega \times (-T_1,0))$.

Proof First by using Holder inequality and Young inequality,

$$\|u\cdot\nabla u\|_{\frac{7}{6}}^{\frac{35}{23}} \leqslant \|\nabla u\|_2^{\frac{35}{23}} \|u\|_{\frac{14}{5}}^{\frac{35}{23}} \leqslant \|\nabla u\|_2^2 + \|u\|_{\frac{14}{5}}^{\frac{70}{11}} \tag{6}$$

In fact, by interpolation inequality,

$$\|u\|_{\frac{14}{5}}^{\frac{70}{11}} \leqslant \|u\|_2^{\frac{30}{11}} \|u\|_4^{\frac{40}{11}}$$

In the following, we show $\nabla p \in L^{\frac{7}{6}}(-T,0,L^{\frac{35}{23}}(\Omega))$

In fact, let $f = \partial_t u - \Delta u$, then first it is obtained that $f \in L^2(-T,0,H_0^{-\frac{5}{2}}(\Omega))$ as mentioned above.

And then we know

$$\begin{cases} \nabla\cdot f = 0, \\ *\mathrm{d}f = *\mathrm{d}(u\cdot\nabla u) \end{cases} \tag{7}$$

in any open set $\Omega \subseteq \mathbf{R}^5$ for a. e. $t \in (-T,0)$.

By the elliptic regularity theory,

$$\|f\|_{\frac{7}{6}}^{\frac{35}{23}} \leqslant \|u\cdot\nabla u\|_{\frac{7}{6}}^{\frac{35}{23}} + \|f\|_{H_0^{-\frac{5}{2}}}^{\frac{35}{23}}$$

By Sobolev embedding, we have $p \in L^{\frac{35}{23}}(-T,0,L^{\frac{35}{23}}(\Omega))$.

Theorem 2 Let (u_n, p_n) is a sequence of weak solutions (1) – (2) in $\Omega \times (-T,0)$ satisfying:

(a) $\sup_{t\in(-T,0)}\int_\Omega |u_n|^4 dx \leqslant E$;

(b) $\int_{\Omega\times(-T,0)} |\nabla u_n|^2 dxdt \leqslant E_1$;

(c) (u_n, p_n) satisfy (4), where E, E_1 some positive constants.

Suppose that (u,p) is a weak limit of (u_n,p_n), then (u,p) is also a suitable weak solution of (1) – (2).

Proof In fact, we can choose a subsequence $\{u_n\}_{n=1}^\infty$ such that

$$u_n \to \hat{u} \text{ weakly in } L^2(-T,0;H^1(\Omega)),$$
$$u_n \to \tilde{u} \text{ weakly } * \text{ in } L^\infty(-T,0;L^4(\Omega)) \quad (8)$$

as $n\to\infty$. Choose $\varphi \in C_0^\infty(\Omega)$ and $\nabla\cdot\varphi = 0$, we have

$$(\partial_t u_n,\varphi) = -(u_n\cdot\nabla u_n,\varphi) - (\nabla u_n,\nabla\varphi) \leqslant \|u_n\|_{L^2}\|\nabla u_n\|_{L^2}\|\varphi\|_{L^\infty} + \|\nabla u_n\|_{L^2}\|\nabla\varphi\|_{L^2} \leqslant (\|u_n\|_{L^2}\|\nabla u_n\|_{L^2} + \|\nabla u_n\|_{L^2})\|\varphi\|_{H^{\frac{5}{2}}}$$

Hence

$$\partial_t u_n \in L^{\frac{7}{6}}(-T,0;H^{-\frac{5}{2}}(\Omega))$$

and $u_n \in C(-T,0;H^{-\frac{5}{2}}(\Omega))$ is uniformly continuous in t.

Claim $\hat{u} = \tilde{u} = u$.

In the following we prove in two steps.

Step 1 It is easy to check that $\hat{u} = \tilde{u}$. In fact, for $\forall w \in L^2(-T,0;H^1(\Omega))$,

$$\int_{-T}^0\int_\Omega (u_n - \hat{u}, w)dxdt \to 0 \ as \ n\to\infty$$

and for $\forall w \in L^1(-T,0;L^{\frac{4}{3}}(\Omega))$,

$$\int_{-T}^0\int_\Omega [u_n - \hat{u}, w]dxdt \to 0 \ as \quad n\to\infty$$

Since $L^2(-T,0;H^1(\Omega)) \subseteq L^1(-T,0;L^{\frac{4}{3}}(\Omega))$ and the uniqueness of Limit,

$$\int_{-T}^0\int_\Omega [\tilde{u} - \hat{u}, w]dxdt \to 0$$

for $\forall w \in L^2(-T,0;H^1(\Omega))$. This implies $\hat{u} = \tilde{u}$ a.e. in $L^2(-T,0;H^1(\Omega)) \cap L^\infty(-T,0;L^4(\Omega))$.

Step 2 $u_n \in C(-T,0;H^{-\frac{5}{2}}(\Omega))$ is uniformly continuous in t, for $\forall t_0 \in (-T,0)$, we can show that $u(x,t_0)$ is well-defined in $L^2(\Omega)$.

Indeed, for any, $\varphi \in C_0^2(\Omega), \nabla\cdot\varphi = 0$, let $\eta(t) \geqslant 0$ is smooth in a δ neighborhood of t_0 (with loss of generality, denote $B_\delta(t_0)$) and has a compact support with $\int_{-T}^0 \eta(t)dt = 1$,

$$\int_{-T}^0\int_\Omega \varphi(x)u(x,t)\eta(t)dxdt = \int_{-T}^0\int_\Omega \varphi(x)u(x,t_0)\eta(t)dxdt + o(1)$$
$$\text{as } \delta\to 0, o(1)\to 0$$

And

$$\int_{-T}^0\int_\Omega \varphi(x)u(x,t)\eta(t)dxdt = \int_{-T}^0\int_\Omega \varphi(x)u_n(x,t)\eta(t)dxdt + o(1) \text{ as } n\to 0, o(1)\to 0$$

According to the weak continuous in t,

$$\int_{-T}^0\int_\Omega \varphi(x)u_n(x,t)\eta(t)dxdt = \int_{-T}^0\int_\Omega \varphi(x)u_n(x,t_0)\eta(t)dxdt + o(1)$$

as $\delta\to 0, o(1)\to 0$ is independent of n.

Hence,

$$\int_\Omega \varphi(x)u(x,t_0)dx \to \int_\Omega \varphi(x)\hat{u}(x,t_0)dx$$

Finally by Theorem 1,

$$u_n \to u \quad (9)$$

converges strongly in $L^2(\Omega\times(-T,0))$. Also, $u \in L^{4,\infty}(\Omega\times(-T,0))$, by interpolation inequality,

$$\|u_n - u\|_3^3 \leqslant \|u_n - u\|_2\|u_n - u\|_4^2$$

Hence from (9),

$$u_n \to u \quad (10)$$

converges strongly in $L^3(\Omega\times(-T,0))$. Since (u,p) is the weak limit of (u_n,p_n), for any smooth $\varphi > 0$ compactly supported in $\Omega\times(-T,0)$, we have that

$$\int_{-T}^0\int_\Omega \varphi(x)|\nabla u(x,t)|^2 dxdt \leqslant \liminf\int_{-T}^0\int_\Omega \varphi(x)|\nabla u_n(x,t)|^2 dxdt$$

From Lemma 1 and (10), the theorem is proved.

2 The Regularity theorem

Using the compactness theorem in the last section, we show the partial regularity of the weak solutions of (1) - (2). Here we give a result which characterizes Hölder continuous functions by the growth of their local integrals.

Theorem 3 Suppose $u \in L^2(\Omega)$ satisfies

$$\int_{B_r(x)} |u - u_{x,r}|^2 dx \leqslant M^2 r^{5+2\alpha} \tag{11}$$

for any $B_r(x) \subseteq \Omega$ and $\alpha \in (0,1)$, where

$$u_{x,r} = \frac{1}{|B_r(x)|}\int_{B_r(x)} u dy$$

then $u \in C^\alpha(\Omega)$.

Proof Denote $R_0 = \mathrm{dist}(\Omega', \partial\Omega), \Omega' \subseteq \Omega$. For any $x_0 \in \Omega'$ and $0 < r_1 < r_2 < R_0$, we have

$$|u_{x_0,r_1} - u_{x_0,r_2}|^2 \leqslant 2(|u_{x_0,r_1} - u_{x_0,r}|^2 + |u_{x_0,r} - u_{x_0,r_2}|^2)$$

and integrating with respect to x in $B_{r_1}(x_0)$

$$|u_{x_0,r_1} - u_{x_0,r_2}|^2 \leqslant \frac{2}{w_n r^5}\left(\int_{B_{r_1}} |u_{x_0,r_1} - u_{x_0,r}|^2 + \int_{B_{r_2}} |u_{x_0,r} - u_{x_0,r_2}|^2\right)$$

from (11),

$$|u_{x_0,r_1} - u_{x_0,r_2}|^2 \leqslant cM^2 r_1^{-5}(r_1^{2\alpha+5} + r_2^{2\alpha+5}) \tag{12}$$

For any$R \leqslant R_0$, with $r_1 = \frac{R}{2^{i+1}}, r_2 = \frac{R}{2^i}$, we obtain

$$\left| u_{x_0\frac{R}{2^{i+1}}} - u_{x_0,\frac{R}{2^i}} \right|^2 \leqslant c2^{-(i+1)\alpha} MR^\alpha$$

and therefore for $h < k$

$$\left| u_{x_0\frac{R}{2^h}} - u_{x_0,\frac{R}{2^k}} \right|^2 \leqslant c(\alpha)2^{-h\alpha} MR^\alpha$$

This shows that $\{u_{x_0,\frac{R}{2^i}}\}$ is a Cauchy sequence, and its limit $\hat{u}(x_0)$ is independent of R. Thus we get

$$\hat{u}(x_0) = \lim_{r\to 0} u_{x_0,r}$$

with

$$|u_{x_0,r} - \hat{u}(x_0)| \leqslant cMr^\alpha$$

for any $0 < r \leqslant R_0$.

Recall that $u_{x,r}$ converges as $r \to 0$ in $L^1(\Omega)$ to the function u, by the Lebesgue Theorem, so we have $u = \hat{u}$ a. a. and (12) implies that $u_{x,r}$ converges uniformly to $u(x)$ in Ω'. Since $x \to u_{x,r}$ is continuous for any $r > 0$, $u(x)$ is continuous. By (3), we get

$$|u(x)| \leqslant CMR_0^\alpha + |u_{x,R}|$$

for any $x \in \Omega'$ and $R \leqslant R_0$. Hence u is bounded in Ω' with the estimate

$$\sup_{\Omega'} |u| \leqslant c(MR_0^\alpha + \|u\|_{L^2(\Omega)})$$

Finally we prove that u is Hölder continuous. Let $x, y \in \Omega'$ with $R = |x - y| < \frac{R_0}{2}$.

Then we have

$$|u(x) - u(y)| \leqslant |u(x) - u_{x,2R}| + |u(y) - u_{y,2R}| + |u_{y,2R} - u_{x,2R}|$$

The first two terms on the right sides are estimated in (11). For the last term we write

$$|u_{y,2R} - u_{x,2R}| \leqslant |u_{y,2R} - u(\zeta)| + |u_{x,2R} - u(\zeta)|$$

and integrating with respect to ζ over $B_{2R}(x) \cap B_{2R}(y)$, which contains $B_R(x)$, yields

$$|u_{y,2R} - u_{x,2R}|^2 \leqslant cM^2 R^{2\alpha}$$

Therefore, we have

$$|u(y) - u(x)| \leqslant cM|x - y|^\alpha$$

In the following we assume (u,p) is a suitable weak solution of Navier-Stokes equations in $\Omega \times (-T_1, 0)$.

Lemma 2 Suppose (u,p) is a suitable weak solution of (1) – (2), if there exists two positive constant ε_0 such that

$$\int_{\Omega\times(-T_1,0)} (|u|^3 + |p|^{\frac{3}{2}}) dz < \varepsilon \tag{13}$$

and

$$u \in L^{4,\infty}(\Omega \times (-T, 0)) \tag{14}$$

then

$$\theta^{-7}\iint_{Q_\theta}\left[\frac{|u - u_\theta|^3}{\theta^{\alpha_0}} + \frac{|p - p_\theta|^{\frac{3}{2}}}{\theta^{\alpha_0}}\right] dx dt \leqslant \frac{1}{2}\int_{-T_1}^0\int_\Omega (|u|^3 + |p|^{\frac{3}{2}}) dx dt \tag{15}$$

for some positive θ and $\alpha_0 \in \left(0, \frac{1}{2}\right)$, where

$$u_\theta = \theta^{-7}\iint_{Q^\theta} u(y,\tau) dy d\tau, p_\theta = \theta^{-5}\int_{B_\theta\times t} p(y,t) dy$$

for $-\theta^2 \leqslant t \leqslant 0$. Denote $Q_\theta = B_\theta \times (-\theta^2, 0)$.

Proof Suppose that Lemma 2 is false, then there is a subsequence of weak solutions (u_i, p_i) with

$$\varepsilon_i = \|u_i\|_{L^3(Q_1)} + \|p_i\|_{L^{\frac{3}{2}}(Q_1)} \tag{16}$$

where $Q_1 = B_1 \times (-1, 0)$, and such that (15) is not valid for (u_i, p_i). Let

$$v_i = \frac{u_i}{\varepsilon_i}, \tilde{p}_i = \frac{p_i}{\varepsilon_i}$$

then

$$\partial_t v_i - \Delta v_i + \varepsilon_i v_i \cdot \nabla v_i + \nabla \tilde{p}_i = 0 \tag{17}$$

A simple computation verifies that (v_i, p_i) satisfies the conditions of the Theorem 2 and (17). One notices that $\tilde{p}_i$ satisfies

$$\Delta \tilde{p}_i = -\varepsilon_i \mathrm{div}(v_i \cdot \nabla v_i) \tag{18}$$

in Q_1, let $(v(x,t), \tilde{p})$ is weak limit of $(v_i, \tilde{p}_i)$, then Theorem 2 implies that

$$\begin{cases} \partial_t v - \Delta v + \nabla \tilde{p} = 0, \\ \nabla \cdot v = 0, \\ \Delta \tilde{p} = 0 \end{cases}$$

in Q_1. By Fatou Lemma,

$$\iint_{Q_1} |v|^3 \mathrm{d}x\mathrm{d}t < 1, \iint_{Q_1} |\tilde{p}|^{\frac{3}{2}} \mathrm{d}x\mathrm{d}t < 1$$

A standard estimates in PDE yield that v and $\tilde{p}$ are smooth in spatial variable and v is Hölder continuous in time variable with exponent $2\alpha_0$. Thus for $\theta \in (0, \frac{1}{2})$, one has

$$\theta^{-7} \iint_{Q_\theta} |v - v_\theta|^3 \mathrm{d}x\mathrm{d}t \leqslant C\theta^{\alpha_0}$$

Since $u_n \to u$ is strong converge in $L^3(Q)$, we have

$$\theta^{-7} \iint_{Q_\theta} |v_i - v_{i,\theta}|^3 \mathrm{d}x\mathrm{d}t \leqslant C\theta^{\alpha_0} \tag{19}$$

for all sufficiently enough i.

Next we consider $\tilde{p}_i$ by (18), we may write for $t \in (-1,0)$, that

$$\tilde{p}_i(x,t) = h_i(x,t) + g_i(x,t) \tag{20}$$

for $x \in B_{\frac{2}{3}}$.

Here

$$\begin{cases} \Delta g_i = -\varepsilon_i \mathrm{div}(v_i \cdot \nabla v_i), x \in B_{\frac{2}{3}} \\ g_i = 0, x \in \partial B_{\frac{2}{3}} \end{cases} \tag{21}$$

and

$$\begin{cases} \Delta h_i = 0, x \in B_{\frac{2}{3}}; \\ h_i = 0, x \in \partial B_{\frac{2}{3}} \end{cases}$$

Denote

$$\tilde{p}_{i,\theta} = h_{i,\theta}$$

then by Calderon-Zygmund estimate and (20),

$$\int_{B_\theta} |g_i(x)|^{\frac{3}{2}} \mathrm{d}x \leqslant \int_{B_\theta} |v_i(x)|^3 \mathrm{d}x \tag{22}$$

Since h_i is harmonic in $B_{\frac{2}{3}}$, any Sobolev norm h_i in a small ball can be estimates by any of its L^p norm in $B_{\frac{2}{3}}$. Thus, by using Poincáre inequality, one can obtain

$$\int_{B_\theta} |h_i(x) - h_{i,\theta}|^{\frac{3}{2}} \mathrm{d}x \leqslant N\theta^{\frac{3}{2}} \int_{B_{\frac{2}{3}}} |\nabla h_i(x)|^{\frac{3}{2}} \mathrm{d}x \leqslant$$
$$N\theta^{5+\frac{3}{2}} \int_{B_{\frac{2}{3}}} |h_i(x) - h_{i,\theta}|^{\frac{3}{2}} \mathrm{d}x \leqslant$$
$$N\theta^{5+\frac{3}{2}} \int_{B_{\frac{2}{3}}} |\tilde{p}_i(x) - h_{i,\theta}|^{\frac{3}{2}} \mathrm{d}x + N\theta^{5+\frac{3}{2}} \int_{B_{\frac{2}{3}}} |g_i(x)|^{\frac{3}{2}} \mathrm{d}x$$

Hence from (20), (22), (23), we get

$$\int_{B_\theta} |\tilde{p}_i(x) - \tilde{p}_{i,\theta}|^{\frac{3}{2}} \mathrm{d}x \leqslant N\theta^{5+\frac{3}{2}} \int_{B_{\frac{2}{3}}} |\tilde{p}_i(x) - h_{i,\theta}|^{\frac{3}{2}} \mathrm{d}x +$$
$$N\theta^{5+\frac{3}{2}} \int_{B_{\frac{2}{3}}} |g_i(x)|^{\frac{3}{2}} \mathrm{d}x \leqslant C\theta^{5+\frac{3}{2}} \tag{24}$$

It is obvious from (24) that

$$\theta^{-7} \int_{Q_\theta}\int |\tilde{p}_i(x) - \tilde{p}_{i,\theta}|^{\frac{3}{2}} \mathrm{d}x \leqslant C\theta^{\alpha_0} \tag{25}$$

for suitable $\theta \in (0, \frac{1}{2})$ and for all sufficiently large i.

Combining (19) and (25), we obtain a contraction and the lemma is proved.

Theorem 4 Under the assumptions of Lemma 2, then for any number k, $\nabla^{k-1} u$ is Hölder continuous in subset $K \subset\subset \Omega \times (-T, 0)$ and the following bound is valid:

$$\max_{z \in K} u < c_0$$

where c_0 is a constant only depending on k.

Proof Let (u,p) be a suitable weak solution such that

$$\int_{Q_1} |u|^3 + |p|^{\frac{3}{2}} \mathrm{d}x\mathrm{d}t < \varepsilon_0$$

Let

$$u_1(x,t) = \frac{u - u_\theta}{\theta^{\frac{\alpha_0}{3}}}(\theta x, \theta^2 t),$$

$$P_1 = \frac{p - p_\theta}{\theta^{\frac{\alpha_0 - 1}{3}}}(\theta x, \theta^2 t)$$

A simple computation yields that is (u_1, p_1) a suitable weak solution of

$$\partial_t u_1 - \Delta u_1 + \theta(u_\theta + u_1 \theta^{\frac{\alpha_0}{3}}) \nabla u_1 + \nabla p_1 = 0$$

Moreover, Lemma 2 implies that

$$\int_{Q_1} |u|^3 + |p|^{\frac{3}{2}} \mathrm{d}x\mathrm{d}t < \varepsilon_0,$$

$$\theta^{-7} \iint_{Q_\theta} \left[\frac{|u_1 - u_{1,\theta}|^3}{\theta^{\alpha_0}} + \frac{|p_1 - p_{1,\theta}|^3}{\theta^{\alpha_0}} \right] \mathrm{d}x\mathrm{d}t \leqslant$$
$$\frac{1}{2} \int_{-T_1}^0 \int_\Omega (|u_1|^3 + |p_1|^{\frac{3}{2}}) \mathrm{d}x\mathrm{d}t \leqslant C\varepsilon_0$$

We repeat the same arguments as Lemma 2, it is concluded that

$$r^{-7} \iint_{Q_r} |u - u_r|^3 \mathrm{d}x\mathrm{d}t \leqslant C\varepsilon_0 r^{\alpha_0}$$

for all $r \in \left(0, \frac{1}{2}\right)$. For all $k = 1, 2, \cdots$, Hölder continuity of u on the set $Q_{\frac{1}{4}}$ follows from Campanato's condition. Moreover, the quantity

$$\sup_{z \in Q_{\frac{1}{4}}} |u(z)|$$

is bounded by an absolute constant.

The case $k > 1$ is treated with the help of the regularity theory for the Stokes equations and bootstrap arguments.

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