第 59 卷　第 1 期　　中山大学学报（自然科学版）　　Vol. 59　No. 1
2020 年　1 月　　ACTA　SCIENTIARUM　NATURALIUM　UNIVERSITATIS　SUNYATSENI　　Jan.　2020

DOI：10. 13471/j. cnki. acta. snus. 2020. 01. 002

关于平衡对的相对和广义 Tate 上同调*

张春霞
（重庆师范大学数学科学学院，重庆 401331）

摘　要：研究了关于平衡对的相对和广义 Tate 上同调，得到了关于平衡对的 Avramov-Martsinkovsky 型正合序列。

关键词：平衡对；相对上同调；广义 Tate 上同调；Avramov-Martsinkovsky 型正合序列

中图分类号：O154. 2；O153. 3　　**文献标志码：**A　　**文章编号：**0529 – 6579（2020）01 – 0009 – 06

Relative and generalized Tate cohomology with respect to balanced pairs

ZHANG Chunxia
（School of Mathematical Sciences，Chongqing Normal University，Chongqing 401331，China）

Abstract：The relative and generalized Tate cohomology with respect to balanced pairs are studied. An Avramov-Martsinkovsky type exact sequence is obtained.

Key words：balanced pair；relative cohomology；generalized Tate cohomology；Avramov-Martsinkovsky type exact sequence

Throughout this paper, $\mathscr{A}$ is an abelian category with enough projectives and injectives. We use $P(\mathscr{A})$ and $I(\mathscr{A})$ to denote the full subcategories of $\mathscr{A}$ consisting of projectives and injectives respectively. Let $\mathscr{X}\subseteq\mathscr{A}$ be a full additive subcategory which is closed under isomorphisms and direct summands.

The concept of balanced pairs was introduced by Chen[1]. This generalizes the notions of right and left balanced functors, which were introduced by Carten and Eilenberg and were generalized to relative homological algebra by Enochs and Jenda[2]. In fact, these theories are based on the classical balanced pair $(P(\mathscr{A}), I(\mathscr{A}))$. Chen showed that for a balanced pair $(\mathscr{X},\mathscr{Y})$ of $\mathscr{A}$, it inherits some nice properties from the classical one[1]. For some work on the balanced pairs please see [1, 3 – 5].

It is well known that derived functor is a powerful tool in studying homological properties of rings and modules in classical homological algebra. For the classical right derived functor $\mathrm{Ext}_{\mathscr{A}}(-,-)$ induced by $\mathrm{Hom}_{\mathscr{A}}(-,-)$ simultaneously measures unprojectiveness of the first variable and uninjectiveness of the second one. So we may say the derived functor $\mathrm{Ext}_{\mathscr{A}}(-,-)$ is based on the classical balanced pair $(P(\mathscr{A}), I(\mathscr{A}))$.

The relative homological algebra, especially Gorenstein homological algebra, as a generalization of the classical one, was developed by Enochs and Jenda[2]. It has been developed to an advanced level in recent year and used in the representation theory of algebras and algebraic geometry. In particular, Enochs and Jenda introduced and studied the Gorenstein right derived functor $\mathrm{Gext}_{\mathscr{A}}(-,-)$ over a Gorenstein ring.

The subject of relative and Tate cohomology theories goes back to Avramov and Martsinkovsky[6]. They studied the theories in the subcategory of modules of finite G – dimension and made an intensive study of the interaction between the three cohomology theories: i. e. the absolute, the relative and the Tate cohomology theories, and conse-

---

* **收稿日期：**2019 – 01 – 26
基金项目：国家自然科学基金（11871125）；重庆市科委项目（cstc 2017jcyjAX0298）
作者简介：张春霞（1979 年生），女；**研究方向：**环的同调理论；E-mail：20160046@ cqnu. edu. cn

quently an exact sequence connecting the absolute cohomology functor Ext, relative cohomology functor Ext_G and Tate cohomology functor $\widehat{\mathrm{Ext}}$ was given as following:

$$0 \to \mathrm{Ext}_G^1(M,N) \to \mathrm{Ext}_R^1(M,N) \to \widehat{\mathrm{Ext}}_R^1(M,N) \to \mathrm{Ext}_G^2(M,N) \to \cdots$$

In [7], Iacob gave another way to prove the existence of above exact sequence, and Tate cohomology in the sense of Iacob is more general.

The aim of this paper is to introduce and study the relative and Iacob's generalized Tate cohomology with respect to balanced pairs. Furthermore, we give the Avramov-Martsinkovsky type exact sequence relative to balanced pairs. Then as applications, many results are generalized and improved.

1 Preliminaries

Let $\mathscr{X},\mathscr{Y} \subseteq \mathscr{A}$ be subcategories. Let $\hat{Z}$ be a cochain complex in $\mathscr{A}$. We say that $\hat{Z}$ is right $\mathscr{X}$-acyclic provided that the Hom complexes $\mathrm{Hom}_{\mathscr{A}}(X,\hat{Z})$ are acyclic for all $X \in \mathscr{X}$. Dually we have the notion of left $\mathscr{Y}$-acyclic complexes.

Let $M \in \mathscr{A}$. A morphism $\theta: X \to M$ called a right $\mathscr{X}$-approximation of M, if $X \in \mathscr{X}$ and any morphism from an object in $\mathscr{X}$ to M factors through θ. The subcategory $\mathscr{X}$ is called contravariantly finite if each object in $\mathscr{A}$ has a right $\mathscr{X}$-approximation (see [8, P. 81] and [9, Def. 1.1]). Recall that $\mathscr{X}$ is an admissible contravariantly finite subcategory provided that each right $\mathscr{X}$-approximation is epic. It is equivalent to that any right $\mathscr{X}$-acyclic complex is indeed acyclic. Dually one has the notion of left $\mathscr{Y}$-approximation and then the notions of (coadmissible) covariantly finite subcategory.

Recall that for a contravariantly finite subcategory $\mathscr{X} \subseteq \mathscr{A}$ and an object $M \in \mathscr{A}$, an $\mathscr{X}$-resolution of M is a complex $\cdots \to X^{-2} \xrightarrow{d^{-2}} X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{\varepsilon} M \to 0$ with each $X^i \in \mathscr{X}$ such that it is right $\mathscr{X}$-acyclic; this is equivalent to that each induced morphism $X^{-n} \to \mathrm{Ker}d^{-n+1}$ is a right $\mathscr{X}$-approximation. Sometimes, we denote the $\mathscr{X}$-resolution by $\widetilde{X} \xrightarrow{\varepsilon} M$ where $\widetilde{X} = \cdots \to X^{-2} \xrightarrow{d^{-2}} X^{-1} \xrightarrow{d^{-1}} X^0 \to 0$ is the deleted $\mathscr{X}$-resolution of M. Note that by a version of comparison Theorem, the $\mathscr{X}$-resolution is unique up to homotopy [2, P. 169, Ex. 2]. Dually one has the notion of $\mathscr{Y}$-coresolution of an object N in a covariantly finite subcategory $\mathscr{Y} \subseteq \mathscr{A}$.

Definition 1 ([1, Def. 1.1]) A pair $(\mathscr{X},\mathscr{Y})$ of additive subcategories in $\mathscr{A}$ is called a balanced pair if the following conditions are satisfied:

(BP0) the subcategory $\mathscr{X}$ is contravariantly finite and $\mathscr{Y}$ is covariantly finite;

(BP1) for each object $M \in \mathscr{A}$, there is an $\mathscr{X}$-resolution $\widetilde{X} \xrightarrow{\varepsilon} M$ such that it is left $\mathscr{Y}$-acyclic;

(BP2) for each object $N \in \mathscr{A}$, there is a $\mathscr{Y}$-coresolution $N \xrightarrow{\eta} \widetilde{Y}$ such that it is right $\mathscr{X}$-acyclic.

It follows from [1, Cor. 2.3], for a balanced pair $(\mathscr{X},\mathscr{Y})$, then $\mathscr{X}$ is admissible if and only if $\mathscr{Y}$ is coadmissible. In this case, we say that the balanced pair $(\mathscr{X},\mathscr{Y})$ is admissible.

We list some examples of admissible balanced pairs as follows.

Example 1 (i) It is well known that the pair $(P(\mathscr{A}), I(\mathscr{A}))$ is an admissible balanced pair. Which we call it classical balanced pair.

(ii) Let R be a ring and Mod R the category of left R-modules. By [2, Example 8.3.2], $(PP(R), PI(R))$ is an admissible balanced pair, where $PP(R)$ and $PI(R)$ are the subcategories of Mod R consisting of pure projective modules and pure injective modules respectively.

(iii) Let R be a Gorenstein ring (that is, R is a left and right Noetherian ring with finite left and right self-injective dimensions). Then by [2, Thm. 12.1.4], we have that $(GP(R), GI(R))$ is an admissible balanced pair in Mod R, where $GP(R)$ and $GI(R)$ are the subcategories of Mod$R$ consisting of Gorenstein projective and Gorenstein injective modules respectively (see [9 – 10]).

(iv) Let R be an FC ring (that is, R is a left and right coherent ring with finite left and right self-FP-injective dimensions). Then by [11, Thm. 3.6], we have that $(DP(R), DI(R))$ is an admissible balanced pair in Mod$R$, where $DP(R)$ and $DI(R)$ are the subcategories of Mod R consisting of Dingprojective and Ding injective modules respectively (see [12-14]).

2 Relative cohomology with respect to balanced pairs

By [1, Lem. 2.1], balanced pairs enjoy certain "balanced" property, that is, let $(\mathscr{X},\mathscr{Y})$ be a balanced pair in $\mathscr{A}$ and $M,N\in\mathscr{A}$ with an $\mathscr{X}$-resolution $\widetilde{X}\xrightarrow{\varepsilon}M$ and a $\mathscr{Y}$-coresolution $N\xrightarrow{\eta}\widetilde{Y}$. Then for each $n\in\mathbf{Z}$ there exists a natural isomorphism

$$\mathrm{H}^n(\mathrm{Hom}_{\mathscr{A}}(\widetilde{X},N))\cong\mathrm{H}^n(\mathrm{Hom}_{\mathscr{A}}(M,\widetilde{Y}))$$

They are independent of the choices of the X-resolutions of M and the Y-coresolutions of N respectively. For any $n\in\mathbf{Z}$, we write

$$\mathrm{Ext}^n_{(\mathscr{X},\mathscr{Y})}(M,N):=\mathrm{H}^n(\mathrm{Hom}_{\mathscr{A}}(\widetilde{X},N))\cong\mathrm{H}^n(\mathrm{Hom}_{\mathscr{A}}(M,\widetilde{Y}))$$

and call it the relative cohomology group with respect to the balanced pair $(\mathscr{X},\mathscr{Y})$.

By Example 1, we have the following

Remark 1 (i) If $(\mathscr{X},\mathscr{Y})=(P(\mathscr{A}),I(\mathscr{A}))$, then $\mathrm{Ext}^n_{(\mathscr{X},\mathscr{Y})}(M,N)$ is exactly the absolute cohomology group $\mathrm{Ext}^n_A(M,N)$.

(ii) Let R be a ring. If $(\mathscr{X},\mathscr{Y})=(PP(R),PI(R))$, then $\mathrm{Ext}^n_{(\mathscr{X},\mathscr{Y})}(M,N)$ coincides with the relative cohomology group $\mathrm{Pext}^n_R(M,N)$ defined in [15-16].

(iii) Let R be a Gorenstein ring. If $(\mathscr{X},\mathscr{Y})=(GP(R),GI(R))$, then $\mathrm{Ext}^n_{(\mathscr{X},\mathscr{Y})}(M,N)$ coincides with the relative cohomology group $\mathrm{Gext}^n_R(M,N)$ defined in [2, P. 296].

(iv) Let R be an FC ring. If $(\mathscr{X},\mathscr{Y})=(DP(R),DI(R))$, then $\mathrm{Ext}^n_{(\mathscr{X},\mathscr{Y})}(M,N)$ coincides with the relative cohomology group $\mathrm{Dext}^n_R(M,N)$ defined in [11, Remark. 3.7].

Then it is easy to check the following results of $\mathrm{Ext}_{(\mathscr{X},\mathscr{Y})}(-,-)$.

Proposition 1 Let $(\mathscr{X},\mathscr{Y})$ be an admissible balanced pair in $\mathscr{A}$ and $M,N\in\mathscr{A}$.

(i) For any $n<0$, $\mathrm{Ext}^n_{(\mathscr{X},\mathscr{Y})}(M,N)=0$ and there exists a natural equivalence $\mathrm{Ext}^0_{(\mathscr{X},\mathscr{Y})}(M,N)\cong\mathrm{Hom}_{\mathscr{A}}(M,N)$.

(ii) $\mathrm{Ext}^n_{(\mathscr{X},\mathscr{Y})}(M,N)=0$ for any $n\in\mathbf{Z}$ if either $M\in\mathscr{X}$ or $N\in\mathscr{Y}$.

(iii) For any right $\mathscr{X}$-acyclic (equivalent left $\mathscr{Y}$-acyclic) complex $0\to A\to B\to C\to 0$ in $\mathscr{A}$, there exist the following long exact sequences

$$\cdots\to\mathrm{Ext}^i_{(\mathscr{X},\mathscr{Y})}(C,N)\to\mathrm{Ext}^i_{(\mathscr{X},\mathscr{Y})}(B,N)\to\mathrm{Ext}^i_{(\mathscr{X},\mathscr{Y})}(A,N)\to\mathrm{Ext}^{i+1}_{(\mathscr{X},\mathscr{Y})}(C,N)\to\cdots$$

and

$$\cdots\to\mathrm{Ext}^i_{(\mathscr{X},\mathscr{Y})}(M,A)\to\mathrm{Ext}^i_{(\mathscr{X},\mathscr{Y})}(M,B)\to\mathrm{Ext}^i_{(\mathscr{X},\mathscr{Y})}(M,C)\to\mathrm{Ext}^{i+1}_{(\mathscr{X},\mathscr{Y})}(M,A)\to\cdots$$

In the following, we want to compare $\mathrm{Ext}_{(\mathscr{X},\mathscr{Y})}$ with the classical $\mathrm{Ext}_{\mathscr{A}}$.

Proposition 2 Let $(\mathscr{X},\mathscr{Y})$ be an admissible balanced pair in $\mathscr{A}$ and $M,N\in\mathscr{A}$. There are natural isomorphisms $\mathrm{Ext}^n_{(\mathscr{X},\mathscr{Y})}(M,N)\cong\mathrm{Ext}^n_{\mathscr{A}}(M,N)$ provided $\mathrm{pd}_A M<\infty$ or $\mathrm{id}_A N<\infty$.

Proof. Assume that $\mathrm{pd}_A M<\infty$, and pick any projective resolution $\widetilde{P}\to M$ of M. Since $\widetilde{P}$ is also an $\mathscr{X}$-resolution of M, and thus

$$\mathrm{Ext}^n_{(\mathscr{X},\mathscr{Y})}(M,N)=\mathrm{H}^n(\mathrm{Hom}_{\mathscr{A}}(\widetilde{P},N))\cong\mathrm{Ext}^n_{\mathscr{A}}(M,N)$$

In the case $\mathrm{id}_A N<\infty$, the proof is similar.

Let $\mathscr{X}$ be a subcategory of $\mathscr{A}$. It is known that $K^*(\mathscr{A})$ is a triangulated category for $*\in\{\text{blank},-,+,b\}$. Denote by $K^*_{R\mathscr{X}\text{-}ac}(\mathscr{A})$ (resp. $K^*_{L\mathscr{X}\text{-}ac}(\mathscr{A})$) full triangulated subcategory of $K^*(\mathscr{A})$ consisting of right $\mathscr{X}$-acyclic (resp. left $\mathscr{X}$-*acyclic*) complexes. Both of them are thick subcategories because they are closed under direct summands. Denote by $\sum^{*}_{R\mathscr{X}\text{-}ac}$ (resp. $\sum^{*}_{L\mathscr{X}\text{-}ac}(\mathscr{A})$) the class of all right (resp. left) $\mathscr{X}$-quasi-isomorphisms in $K^{*}(\mathscr{A})$. Then a cochain map is a right (resp. left) $\mathscr{X}$-quasi-isomorphism if and only if its mapping cone is right (resp. left) $\mathscr{X}$-acyclic. Thus $\sum^{*}_{R\mathscr{X}\text{-}ac}(\mathscr{A})$ (resp. $\sum^{*}_{L\mathscr{X}\text{-}ac}(\mathscr{A})$) is the saturated compatible multiplicative system determinated by $K^{*}_{R\mathscr{X}\text{-}ac}(\mathscr{A})$ (resp. $K^{*}_{L\mathscr{X}\text{-}ac}(\mathscr{A})$).

By [3], the vedier quotient category $D^{*}_{R\mathscr{X}}(\mathscr{A}) = K^{*}(\mathscr{A})/K^{*}_{R\mathscr{X}\text{-}ac}(\mathscr{A})$ is called the right $\mathscr{X}$-derived category of $\mathscr{A}$, where $* \in \{\text{blank}, -, +, b\}$. The left $\mathscr{X}$-derived category $D^{*}_{L\mathscr{X}}(\mathscr{A})$ of A is defined dually.

For a balanced pair $(\mathscr{X},\mathscr{Y})$ in $\mathscr{A}$, it follows from [3, Prop. 4.5], we have a triangle – equivalence$D^{b}_{R\mathscr{X}}(\mathscr{A}) = D^{b}_{L\mathscr{X}}(\mathscr{A})$, both of them we call the relative bounded derived category with respect to the balanced pair $(\mathscr{X},\mathscr{Y})$, and denote them by $D^{b}_{(\mathscr{X},\mathscr{Y})}(\mathscr{A})$.

The next result has been proved by Li, Wang and Huang [3, Prop. 4.6], which means that the relative cohomology group $\mathrm{Ext}^{n}_{(\mathscr{X},\mathscr{Y})}(M,N)$ may be computed in the relative bounded derived category $D^{b}_{(\mathscr{X},\mathscr{Y})}(\mathscr{A})$.

Proposition 3 Let $(\mathscr{X},\mathscr{Y})$ be a balanced pair in $\mathscr{A}$. Then for any $M,N \in \mathscr{A}$ and $i \geqslant 1$, there exists an isomorphism of abelian groups:

$$\mathrm{Ext}^{n}_{(\mathscr{X},\mathscr{Y})}(M,N) \cong \mathrm{Hom}\, D^{b}_{(\mathscr{X},\mathscr{Y})}(\mathscr{A})(M,N[i])$$

3 Generalized Tate cohomology relative to balanced pairs

Let R be a ring. Recall that for any R-module M with finite Gorenstein projective dimension, there exists a complete resolution, that is, a diagram $T \xrightarrow{\vartheta} P \xrightarrow{\pi} M$ where $P \xrightarrow{\pi} M$ is a projective resolution, T is a totally acyclic complex of projective modules, and ϑ_i is bijective for all $i \gg 0$. For any R – module N and each $n \in \mathbf{Z}$, the nth Tate cohomology group is defined as

$$\widehat{\mathrm{Ext}}^{n}_{R}(M,N) = \mathrm{H}^{n}(\mathrm{Hom}_{R}(T,N))$$

which is independent of choices of resolutions and liftings.

The close relations between absolute, relative and Tate cohomology are illuminated by an Avramov – Martsinkovsky type exact sequence (see [6, Thm. 7.1]) as follows:

$$0 \to \mathrm{Ext}^{1}_{GP}(M,N) \to \mathrm{Ext}^{1}_{R}(M,N) \to \widehat{\mathrm{Ext}}^{1}_{R}(M,N) \to \mathrm{Ext}^{2}_{GP}(M,N) \to \cdots$$
$$\to \mathrm{Ext}^{i}_{GP}(M,N) \to \mathrm{Ext}^{i}_{R}(M,N) \to \widehat{\mathrm{Ext}}^{i}_{R}(M,N) \to \mathrm{Ext}^{i+1}_{GP}(M,N) \to \cdots$$

In [7], Iacob gave another way to prove the existence of above exact sequence: Let $\widetilde{P} \to M \to 0$ and $\widetilde{G} \to M \to 0$ be a projective and a GP-resolutionof M respectively. Then there exists a morphism$f$: $\widetilde{P} \to \widetilde{G}$. She argued that if M has finite Gorenstein projective dimension, then $\widehat{\mathrm{Ext}}^{n}_{R}(M,N)$ is exactly $\mathrm{H}^{n+1}(\mathrm{Hom}_{R}(\mathrm{Cone}(f),N))$.

Obviously, Tate cohomology in the sense of Iacob is more general. In this section we consider Iacob's generalized Tate cohomology relative to balanced pairs.

Definition 2 Let $(\mathscr{X},\mathscr{Y})$ be an admissible balanced pair in$\mathscr{A}$ and $M,N \in \mathscr{A}$.

(i) Choosing a projective resolution $\widetilde{P} \to M \to 0$ and an $\mathscr{X}$-resolution $\widetilde{X} \to M \to 0$ of M respectively. Then there exists a morphism$f$: $\widetilde{P} \to \widetilde{X}$. Define a relative Generalized Tate cohomology as

$$\overline{\mathrm{Ext}}^{n}_{(\mathscr{X},\mathscr{Y})}(M,N) = \mathrm{H}^{n+1}(\mathrm{Hom}_{\mathscr{A}}(\mathrm{Cone}(f),N))$$

(ii) Choosing an injective coresolution $0 \to N \to \widetilde{I}$ and a Y – coresolution $0 \to N \to \widetilde{Y}$ of$N$ respectively. Then there exists a morphism g: $\widetilde{Y} \to \widetilde{I}$. Define a relative Generalized Tate cohomology as

$$\underline{\mathrm{Ext}}^{n}_{(\mathscr{X},\mathscr{Y})}(M,N) = \mathrm{H}^{n}(\mathrm{Hom}_{\mathscr{A}}(M,\mathrm{Cone}(g)))$$

Remark 2 It follows from [1, Prop. 2.6] and [7, Thm. 1], for any $n\in\mathbf{Z}$, there exists an isomorphism of abelian groups

$$\overline{\mathrm{Ext}}^n_{(\mathscr{X},\mathscr{Y})}(M,N)\cong\underline{\mathrm{Ext}}^n_{(\mathscr{X},\mathscr{Y})}(M,N)$$

We denote both abelian groups by $\widehat{\mathrm{Ext}}^n_{(\mathscr{X},\mathscr{Y})}(M,N)$, and call it the relative Generalized Tate cohomology with respect to the balanced pair $(\mathscr{X},\mathscr{Y})$.

Associate Example 1, we have the following

Example 2 (i) Let R be a Gorenstein ring. If $(\mathscr{X},\mathscr{Y})=(GP(R),GI(R))$, then $\widehat{\mathrm{Ext}}^n_{(\mathscr{X},\mathscr{Y})}(M,N)$ coincides with the usual Tate cohomology $\widehat{\mathrm{Ext}}^n_R(M,N)\cong\overline{\mathrm{Ext}}^n_R(M,N)$ in [17, Thm. 2], which were defined by means of complete resolutions (see [6]).

(ii) Let R be an FC ring. If $(\mathscr{X},\mathscr{Y})=(DP(R),DI(R))$, then $\widehat{\mathrm{Ext}}^n_{(\mathscr{X},\mathscr{Y})}(M,N)$ coincides with $\widehat{\mathrm{Ext}}^n_{DP}(M,N)\cong\widehat{\mathrm{Ext}}^n_{DI}(M,N)$ in [18, Prop. 5.5]. Both of them are identical with the Tate cohomology $\widehat{\mathrm{Dext}}^n_R(M,N)$ defined in [19, Thm. 5.2], which was defined by means of Tate $\mathscr{F}$-resolutions (see [19, Remark. 4.13]).

It follows from [7], we have the following results of long exact sequences about $\widehat{\mathrm{Ext}}_{(\mathscr{X},\mathscr{Y})}(-,-)$.

Proposition 4 Let $(\mathscr{X},\mathscr{Y})$ be an admissible balanced pair in $\mathscr{A}$ and $M,N\in\mathscr{A}$. If $0\to A\to B\to C\to 0$ is a right $\mathscr{X}$-acyclic (equivalent left $\mathscr{Y}$-acyclic) complex in $\mathscr{A}$, then there are exact sequences

$$\cdots\to\widehat{\mathrm{Ext}}^i_{(\mathscr{X},\mathscr{Y})}(C,N)\to\widehat{\mathrm{Ext}}^i_{(\mathscr{X},\mathscr{Y})}(B,N)\to\widehat{\mathrm{Ext}}^i_{(\mathscr{X},\mathscr{Y})}(A,N)\to\widehat{\mathrm{Ext}}^{i+1}_{(\mathscr{X},\mathscr{Y})}(C,N)\to\cdots$$

and

$$\cdots\to\widehat{\mathrm{Ext}}^i_{(\mathscr{X},\mathscr{Y})}(M,A)\to\widehat{Ext}^i_{(\mathscr{X},\mathscr{Y})}(M,B)\to\widehat{\mathrm{Ext}}^i_{(\mathscr{X},\mathscr{Y})}(M,C)\to\widehat{\mathrm{Ext}}^{i+1}_{(\mathscr{X},\mathscr{Y})}(M,A)\to\cdots$$

4 Avramov-Martsinkovsky type exact sequence

In this section, for a balanced pair $(\mathscr{X},\mathscr{Y})$ in $\mathscr{A}$, we give an Avramov-Martsinkovsky type exact sequence with $\widehat{\mathrm{Ext}}_{(\mathscr{X},\mathscr{Y})}$, $\mathrm{Ext}_{(\mathscr{X},\mathscr{Y})}$ and $\mathrm{Ext}_{\mathscr{A}}$. Our proofs using relative derived categories are different form [6] and our results improve that [7, 17-19].

Theorem 1 Let $(\mathscr{X},\mathscr{Y})$ be an admissible balanced pair in $\mathscr{A}$ and $M,N\in\mathscr{A}$. Then there is an exact sequence

$$\begin{aligned}0&\to\mathrm{Ext}^1_{(\mathscr{X},\mathscr{Y})}(M,N)\to\mathrm{Ext}^1_{\mathscr{A}}(M,N)\to\widehat{\mathrm{Ext}}^1_{(\mathscr{X},\mathscr{Y})}(M,N)\to\mathrm{Ext}^2_{(\mathscr{X},\mathscr{Y})}(M,N)\to\cdots\\&\to\mathrm{Ext}^i_{(\mathscr{X},\mathscr{Y})}(M,N)\to\mathrm{Ext}^i_{\mathscr{A}}(M,N)\to\widehat{\mathrm{Ext}}^i_{(\mathscr{X},\mathscr{Y})}(M,N)\to\mathrm{Ext}^{i+1}_{(\mathscr{X},\mathscr{Y})}(M,N)\to\cdots\end{aligned}$$

Proof. Take a projective resolution $\widetilde{P}\to M\to 0$ and an $\mathscr{X}$-resolution $\widetilde{X}\to M\to 0$ of M respectively. For a morphism $f:\widetilde{P}\to\widetilde{X}$, there is a distinguished triangle

$$\widetilde{P}\xrightarrow{f}\widetilde{X}\to\mathrm{Cone}(f)\to\widetilde{P}[1]$$

in $K(\mathscr{A})$. By applying $\mathrm{Hom}_{K(\mathscr{A})}(-,N)$ to it, we get an exact sequence

$$\begin{aligned}\cdots&\to\mathrm{Hom}_{K(\mathscr{A})}(\widetilde{P}[-n+1],N)\to\mathrm{Hom}_{K(\mathscr{A})}(\mathrm{Cone}(f)[-n],N)\\&\to\mathrm{Hom}_{K(\mathscr{A})}(\widetilde{X}[-n],N)\to\mathrm{Hom}_{K(\mathscr{A})}(\widetilde{P}[-n],N)\\&\to\mathrm{Hom}_{K(\mathscr{A})}(\mathrm{Cone}(f)[-n-1],N)\to\mathrm{Hom}_{K(\mathscr{A})}(\widetilde{X}[-n-1],N)\to\cdots\end{aligned}$$

It follows from Proposition 3 that

$$\begin{aligned}\mathrm{Hom}_{K(\mathscr{A})}(\widetilde{X}[-n],N)&\cong\mathrm{Hom}_{D_{R\mathscr{X}}(\mathscr{A})}(\widetilde{X}[-n],N)\\&\cong\mathrm{Hom}_{D^b_{(\mathscr{X},\mathscr{Y})}(\mathscr{A})}(M,N[n])\end{aligned}$$

$$\cong \mathrm{Ext}^{n}_{(\mathscr{X},\mathscr{Y})}(M,N)$$

Note that $\mathrm{Hom}_{K(\mathscr{A})}(\widetilde{P}[-n],N) = \mathrm{Ext}^{n}_{\mathscr{A}}(M,N)$, and it is easily seen that left exactness of Hom and the long exact sequence shows that $\widehat{\mathrm{Ext}}^{n}_{(\mathscr{X},\mathscr{Y})}(M,N)$ vanishes when ever $n < 1$. Then we have the desired exact sequence.

In the following, we give some applications of Theorem 1.

Corollary 1 Let R be a Gorenstein ring. For each R-modules M and N, there is an exact sequence

$$0 \to \mathrm{Gext}^{1}_{R}(M,N) \to \mathrm{Ext}^{1}_{R}(M,N) \to \widehat{\mathrm{Ext}}^{1}_{R}(M,N) \to \mathrm{Gext}^{2}_{R}(M,N) \to \cdots$$

$$\to \mathrm{Gext}^{i}_{R}(M,N) \to \mathrm{Ext}^{i}_{R}(M,N) \to \widehat{\mathrm{Ext}}^{i}_{R}(M,N) \to \mathrm{Gext}^{i+1}_{R}(M,N) \to \cdots$$

Corollary 2 Let R be an FC ring. For each R-modules M and N, there is an exact sequence

$$0 \to \mathrm{Dext}^{1}_{R}(M,N) \to \mathrm{Ext}^{1}_{R}(M,N) \to \widehat{\mathrm{Dext}}^{1}_{R}(M,N) \to \mathrm{Dext}^{2}_{R}(M,N) \to \cdots$$

$$\to \mathrm{Dext}^{i}_{R}(M,N) \to \mathrm{Ext}^{i}_{R}(M,N) \to \widehat{\mathrm{Dext}}^{i}_{R}(M,N) \to \mathrm{Dext}^{i+1}_{R}(M,N) \to \cdots$$

Recall that a left R-module P is said to be pure projective, if for every pure exact sequence $0 \to T \to N \to N/T \to 0$, $\mathrm{Hom}(P,N) \to \mathrm{Hom}(P,N/T) \to 0$ is exact. Thus projective and finitely presented modules are pure projective.

Corollary 3 Let R be a ring. For each R-modules M and N, there is an exact sequence

$$0 \to \mathrm{Pext}^{1}_{R}(M,N) \to \mathrm{Ext}^{1}_{R}(M,N) \to \widehat{\mathrm{Ext}}^{1}_{(PP(R),PI(R))}(M,N) \to \cdots$$

$$\to \mathrm{Ext}^{i}_{R}(M,N) \to \widehat{\mathrm{Ext}}^{i}_{PP(R),PI(R))}(M,N) \to \mathrm{Pext}^{i+1}_{R}(M,N) \to \cdots$$

References:

[1] CHEN X W. Homotopy equivalences induced by balanced pairs [J]. J Algebra, 2010, 324: 2718 – 2731.

[2] ENOCHS E E, JENDA O M G. Relative homological algebra [M]. New York: Walter De Gruyter, 2000.

[3] LI H H, WANG J F, HUANG Z Y. Applications of balanced pairs [J]. Sci China Math, 2016, 59(5): 861 – 874.

[4] WANG J F, LI H H, HUANG Z Y. Applications of exact structures in abelian categories [J]. Publ Math Debrecen, 2016, 88: 269 – 286.

[5] WANG J F, HUANG Z Y. Relative left derived functors of tensor product functors [J]. Acta Math Sin (Engl Ser), 2016, 32(7): 753 – 764.

[6] AVRAMOV L L, MARTSINKOVSKY A. Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension [J]. Proc London Math Soc, 2002, 85(3): 393 – 440.

[7] IACOB A. Balance in generalized Tate cohomology [J]. Comm Algebra, 2005, 33: 2009 – 2024.

[8] AUSLANDER M, SMAL∅ S O. Preprojective modules over Artin algebras [J]. J Algebra, 1980, 66: 61 – 122.

[9] ENOCHS E E, JENDA O M G. Gorenstein injective and projective modules [J]. Math Z, 1995, 220: 611 – 633.

[10] HOLM H. Gorenstein homological dimensions [J]. J. Pure Appl Algebra, 2004, 189: 167 – 193.

[11] YANG G. Homological properties of modules over Ding-Chen rings [J]. J Korean Math Soc, 2012, 49: 31 – 47.

[12] DING N Q, LI Y L, MAO L X. Strongly Gorenstein flat modules [J]. J Aust Math Soc, 2009, 86(3): 323 – 338.

[13] GILLESPIE J. Model structures on modules over Ding – Chen rings [J]. Homology Homotopy Appl, 2010, 12(1): 61 – 73.

[14] MAO L X, DING N Q. Gorenstein FP – injective and Gorenstein flat modules [J]. J Algebra Appl, 2008, 7: 491 – 506.

[15] KIELPINSKI R, SIMSON D. On pure homological dimension [J]. Bull Acad Pol Sci, 1975, 23: 1 – 6.

[16] SIMSON D. On pure global dimension of locally finitely presented Grothendieck categories [J]. Fund Math, 1977, 96: 91 – 116.

[17] IACOB A. Generalized Tate cohomology [J]. Tsukuba J Math, 2005, 29: 389 – 404.

[18] REN W, LIU Z K, YANG G. Derived categories with respect to Ding modules [J]. J Algebra Appl, 2013, 12: 1350021.

[19] ZHANG C X. Relative and Tate cohomology of Ding modules and complexes [J]. J Korean Math Soc, 2015, 52(4): 821 – 838.

（责任编辑 冯兆永）