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相空间中 Herglotz 型微分变分原理与一类新型绝热不变量*

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摘　要：基于 Herglotz 型微分变分原理，研究了相空间中非保守系统的绝热不变量问题。首先，列写出基于 Herglotz 广义变分原理的 Hamilton 正则方程；其次，基于 Hamilton-Herglotz 作用量在群的无穷小变换下的不变性，给出了相空间中新型精确不变量，并进一步研究在小扰动作用下的摄动，得到了系统的一类新型绝热不变量；再次，给出了逆定理；最后，举例说明结果的应用。

关键词：非保守系统；Herglotz 型微分变分原理；绝热不变量；相空间

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Differential variational principle of Herglotz type and a new type of adiabatic invariants in phase space

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Abstract: According to differential variational principle of Herglotz type, this paper studies the adiabatic invariants for non-conservative system in phase space. Firstly, the Hamilton canonical equations based upon the generalized variational principle of Herglotz are given. Secondly, by using the invariance of the Hamilton-Herglotz action under the infinitesimal transformations, the new type of exact invariants in phase space are established, and the perturbation of the system with the action of small disturbance is investigated, and a new type of adiabatic invariants of the system are obtained. Thirdly, the inverse theorem is given. In the end of the paper, an example is given to illustrate the application of the results.

Key words: non-conservative system; differential variational principle of Herglotz type; adiabatic invariants; phase space

动力学系统的对称性和守恒量具有重要的数学意义和物理意义，关于这方面的研究已经取得了许多重要的结果[1-4]。实际上绝热不变量和守恒量的关系是密不可分的，绝热不变量是一定条件下近似不变的

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量。1996年赵跃宇和梅凤翔在增广相空间中研究了完整和非完整力学系统的绝热不变量及其逆问题[5]，给出了力学系统的 Noether 型绝热不变量。此后，Noether 型绝热不变量被推广到 Birkhoff 系统[6]、Lagrange 系统[7]、准坐标下非完整系统[8]等。张毅给出了广义经典力学[9]、Birkhoff 系统[10]的 Hojman 型绝热不变量。罗绍凯给出了 Lutzky 型绝热不变量[11]。丁宁等给出可控非完整系统 Mei 型绝热不变量[12]。最近，关于分数阶力学系统和非线性非保守系统的绝热不变量研究也取得了一些新进展[13-17]。

众所周知，对于非保守系统或耗散系统，经典变分原理难以给出其一个变分描述。例如：适用于完整保守系统的著名的 Hamilton 原理向非保守系统推广时遇到极大的困难。一般地，完整非保守系统的 Hamilton 原理 $\int_{t_0}^{t}(\delta T+Q_s\delta q_s)\,\mathrm{d}t=0$ 不能表示为某个作用量泛函的变分等于零的形式，因此它不再是一个稳定作用量原理。为此，Herglotz[18]提出了一个变分原理，其作用量泛函 $z(t)$ 是由微分方程 $\dot z(t)=L(t,q_s(t),\dot q_s(t),z(t))$ 来定义的。Herglotz 广义变分原理给出了非保守系统的一个变分描述。当 Lagrange 函数 L 不依赖于作用量 z 时，该原理退化为经典 Hamilton 原理。因此，Herglotz 广义变分原理不仅为研究非保守动力学系统提供了一种有效途径，而且可以统一地处理保守系统和非保守系统的问题。近年来，关于 Herglotz 广义变分原理及其应用研究取得了一些重要成果[19-27]。本文将进一步基于相空间中 Herglotz 型微分变分原理，给出一类新型绝热不变量，证明该绝热不变量存在的条件及其形式，并研究其逆问题。

1 微分变分原理与精确不变量

相空间中 Herglotz 型微分变分原理为[21]：

$$\exp\left(\int_{t_1}^{t}\frac{\partial H}{\partial z}\mathrm{d}\theta\right)\left[\left(-\dot p_s-\frac{\partial H}{\partial q_s}-p_s\frac{\partial H}{\partial z}\right)\delta q_s+\left(\dot q_s-\frac{\partial H}{\partial p_s}\right)\delta p_s\right]=0 \tag{1}$$

其中，$\dot z(t)=p_s(t)\dot q_s(t)-H(t,q_s(t),p_s(t),z(t))$、$q_s(t)$、$p_s(t)$ $(s=1,2,\cdots,n)$ 分别为系统的广义坐标和广义动量，$H(t,q_k,p_k,z)$ 为 Herglotz 意义下的 Hamilton 函数。

对于完整系统，δq_s 和 δp_s 互相独立，因此有

$$\exp\left(\int_{t_1}^{t}\frac{\partial H}{\partial z}\mathrm{d}\theta\right)\left(\dot q_s-\frac{\partial H}{\partial p_s}\right)=0,$$
$$\exp\left(\int_{t_1}^{t}\frac{\partial H}{\partial z}\mathrm{d}\theta\right)\left(-\dot p_s-\frac{\partial H}{\partial q_s}-p_s\frac{\partial H}{\partial z}\right)=0\ (s=1,2,\cdots,n) \tag{2}$$

或

$$\dot q_s=\frac{\partial H}{\partial p_s},$$
$$\dot p_s=-\frac{\partial H}{\partial q_s}-p_s\frac{\partial H}{\partial z}\ (s=1,2,\cdots,n) \tag{3}$$

方程（3）是相空间中非保守系统基于 Herglotz 广义变分原理的 Hamilton 正则方程[21]。

引进时间 t 和广义坐标 q_s 和广义动量 p_s 的参数无穷小变换

$$t^*=t+\Delta t,$$
$$q_s^*(t^*)=q_s(t)+\Delta q_s,$$
$$p_s^*(t^*)=p_s(t)+\Delta p_s(s=1,2,\cdots,n) \tag{4}$$

或其展开式

$$t^*=t+\varepsilon\tau(t,q_k,p_k),q_s^*(t^*)=q_s(t)+\varepsilon\xi_s(t,q_k,p_k),$$
$$p_s^*(t^*)=p_s(t)+\varepsilon\eta_s(t,q_k,p_k)(s,k=1,2,\cdots,n) \tag{5}$$

由等时变分和非等时变分之间的关系可得[21]

$$\delta q_s=\varepsilon[\xi_s^0(t,q_k,\dot q_k)-\dot q_s\tau^0(t,q_k,\dot q_k)]\ (s,k=1,2,\cdots,n) \tag{6}$$

同理

$$\delta p_s=\varepsilon[\eta_s^0(t,q_k,p_k)-\dot p_s\tau^0(t,q_k,p_k)]\ (s,k=1,2,\cdots,n) \tag{7}$$

其中，ε 为无限小参数，τ^0、ξ_s^0 和 η_s^0 称为无穷小变换的生成函数。

将式（6），（7）代入原理（1），整理得

$$\varepsilon\left\{\exp\left(\int_{t_1}^{t}\frac{\partial H}{\partial z}\mathrm{d}\theta\right)\left[\left(-\dot{p}_s-\frac{\partial H}{\partial q_s}-p_s\frac{\partial H}{\partial z}\right)(\xi_s^0-\dot{q}_s\tau^0)+\left(\dot{q}_s-\frac{\partial H}{\partial p_s}\right)(\eta_s^0-\dot{p}_s\tau^0)\right]\right\}=0 \tag{8}$$

由于

$$\frac{\mathrm{d}}{\mathrm{d}t}H=\frac{\partial H}{\partial t}+\frac{\partial H}{\partial q_s}\dot{q}_s+\frac{\partial H}{\partial p_s}\dot{p}_s+\frac{\partial H}{\partial z}(p_s\dot{q}_s-H) \tag{9}$$

在式（8）中加上并减去函数 $\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\left[G^0\exp\left(\int_{t_1}^{t}\frac{\partial H}{\partial z}\mathrm{d}\theta\right)\right]$ 得

$$\begin{aligned}&\varepsilon\left\{\exp\left(\int_{t_1}^{t}\frac{\partial H}{\partial z}\mathrm{d}\theta\right)\left[-\frac{\partial H}{\partial q_s}\xi_s^0+p_s\dot{\xi}_s^0+\left(\dot{q}_s-\frac{\partial H}{\partial p_s}\right)\eta_s^0-\frac{\partial H}{\partial t}\tau^0-H\dot{\tau}^0+\frac{\partial H}{\partial z}G^0+\dot{G}^0\right]\right.\\&\left.-\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left(\int_{t_1}^{t}\frac{\partial H}{\partial z}\mathrm{d}\theta\right)(p_s\xi_s^0-H\tau^0+G^0)\right]\right\}=0\end{aligned} \tag{10}$$

其中，$G^0=G^0(t,q_s,p_s)$ 称为规范函数。式（10）是相空间中非保守系统的 Herglotz 型微分变分原理不变性条件的变换。由式（10），立即可得到

定理 1　对于相空间中非保守系统（2），如果存在规范函数 G^0 使无限小生成元 τ^0，ξ_s^0 和 η_s^0 满足如下条件

$$-\frac{\partial H}{\partial q_s}\xi_s^0+p_s\dot{\xi}_s^0+\left(\dot{q}_s-\frac{\partial H}{\partial p_s}\right)\eta_s^0-\frac{\partial H}{\partial t}\tau^0-H\dot{\tau}^0+\frac{\partial H}{\partial z}G^0+\dot{G}^0=0 \tag{11}$$

则系统存在守恒量

$$I_0=\exp\left(\int_{t_1}^{t}\frac{\partial H}{\partial z}\mathrm{d}\theta\right)(p_s\xi_s^0-H\tau^0+G^0)=\mathrm{const.} \tag{12}$$

当 $G^0\equiv 0$ 时，定理 1 给出文献［20］的结果。守恒量（12）是系统未受扰动时的不变量，因此它是一个精确不变量。

2　一类新型绝热不变量

如果 $I_m(t,q_s,p_s,z,\varepsilon)$ 是相空间中非保守系统的一个含有 ε 的最高次幂为 m 的物理量，它对时间 t 的一阶导数正比于 ε^{m+1}，那么 I_m 称为该系统的 m 阶绝热不变量。

假设相空间中非保守系统（2）受到了一个小扰动 εQ_s 的作用，则方程（2）成为

$$\begin{aligned}&\exp\left(\int_{t_1}^{t}\frac{\partial H}{\partial z}\mathrm{d}\theta\right)\left(\dot{q}_s-\frac{\partial H}{\partial p_s}\right)=0,\\&\exp\left(\int_{t_1}^{t}\frac{\partial H}{\partial z}\mathrm{d}\theta\right)\left(-\dot{p}_s-\frac{\partial H}{\partial q_s}-p_s\frac{\partial H}{\partial q_s}\right)=-\varepsilon Q_s\end{aligned} \tag{13}$$

由于小扰动 εQ_s 的作用，该系统原有的对称性和不变量都会发生改变。假设受扰系统的无限小生成函数 $\tau(t,q_k,p_k)$，$\xi_s(t,q_k,p_k)$ 及 $\eta_s(t,q_k,p_k)$ 可表示为

$$\begin{aligned}\tau&=\tau^0+\varepsilon\tau^1+\varepsilon^2\tau^2+\cdots,\\\xi_s&=\xi_s^0+\varepsilon\xi_s^1+\varepsilon^2\xi_s^2+\cdots,\\\eta_s&=\eta_s^0+\varepsilon\eta_s^1+\varepsilon^2\eta_s^2+\cdots\end{aligned} \tag{14}$$

并满足

$$\exp\left(\int_{t_1}^{t}\frac{\partial H}{\partial z}\mathrm{d}\theta\right)\left[-\frac{\partial H}{\partial q_s}\xi_s+p_s\dot{\xi}_s+\left(\dot{q}_s-\frac{\partial H}{\partial p_s}\right)\eta_s-\frac{\partial H}{\partial t}\tau-H\dot{\tau}+\frac{\partial H}{\partial z}G+\dot{G}\right]+\varepsilon Q_s(\xi_s-\dot{q}_s\tau)=0 \tag{15}$$

其中，G 为规范函数，记为

$$G=G^0+\varepsilon G^1+\varepsilon^2G^2+\cdots \tag{16}$$

定理 2　如果相空间中非保守系统（2）受到小扰动 εQ_s 的作用，若存在规范函数 G^j，使无穷小变换的生成函数 τ^j，ξ_s^j 和 η_s^j 满足

$$\begin{aligned}&\exp\left(\int_{t_1}^{t}\frac{\partial H}{\partial z}\mathrm{d}\theta\right)\left[-\frac{\partial H}{\partial q_s}\xi_s^j+p_s\dot{\xi}_s^j+\left(\dot{q}_s-\frac{\partial H}{\partial p_s}\right)\eta_s^j-\frac{\partial H}{\partial t}\tau^j-H\dot{\tau}^j+\frac{\partial H}{\partial z}G^j+\dot{G}^j\right]\\&+Q_s(\xi_s^{j-1}-\dot{q}_s\tau^{j-1})=0\ (j=0,1,2,\cdots)\end{aligned} \tag{17}$$

则

$$I_m = \sum_{j=0}^{m} \varepsilon^j \exp\left(\int_{t_1}^{t} \frac{\partial H}{\partial z}\mathrm{d}\theta\right)(p_s\xi_s^j - H\tau^j + G^j) \tag{18}$$

是该系统的一个 m 阶绝热不变量。

证明： 由条件（17）和方程（13），得

$$\begin{aligned}\frac{\mathrm{d}I_m}{\mathrm{d}t} &= \sum_{j=0}^{m} \varepsilon^j \exp\left(\int_{t_1}^{t} \frac{\partial H}{\partial z}\mathrm{d}\theta\right)\left[\frac{\partial H}{\partial z}p_s\xi_s^j - \frac{\partial H}{\partial z}H\tau^j + \frac{\partial H}{\partial z}G^j + \dot{p}_s\xi_s^j + p_s\dot{\xi}_s^j - H\dot{\tau}^j - \frac{\partial H}{\partial t}\tau^j - \frac{\partial H}{\partial q_s}\dot{q}_s\tau^j - \frac{\partial H}{\partial z}p_s\dot{q}_s\tau^j + \frac{\partial H}{\partial z}H\tau^j + \dot{G}^j\right] \\ &= \sum_{j=0}^{m} \varepsilon^j \exp\left(\int_{t_1}^{t} \frac{\partial H}{\partial z}\mathrm{d}\theta\right)\left[\left(\frac{\partial H}{\partial z}p_s + \dot{p}_s + \frac{\partial H}{\partial q_s}\right)(\xi_s^j - \dot{q}_s\tau^j)\right. \\ &\quad \left. + \dot{p}_s\dot{q}_s\tau^j - \frac{\partial H}{\partial q_s}\xi_s^j - \frac{\partial H}{\partial p_s}\dot{p}_s\tau^j + p_s\dot{\xi}_s^j - H\dot{\tau}^j - \frac{\partial H}{\partial t}\tau^j + \frac{\partial H}{\partial z}G^j + \dot{G}^j\right] \\ &= \sum_{j=0}^{m} \varepsilon^j[\varepsilon Q_s(\xi_s^j - \dot{q}_s\tau^j) - Q_s(\xi_s^{j-1} - \dot{q}_s\tau^{j-1})] \\ &= \varepsilon^{m+1}Q_s(\xi_s^m - \dot{q}_s\tau^m)\end{aligned} \tag{19}$$

因此，I_m 是一个 m 阶绝热不变量。式（18）是我们基于 Herglotz 广义变分原理导出的一类新型绝热不变量。

3 逆问题

假设相空间中非保守系统（2）在小扰动 εQ_s 作用下存在如下一阶绝热不变量

$$I_1 = F_0(t,q_s,p_s,z) + \varepsilon F_1(t,q_s,p_s,z) \tag{20}$$

因其轨道应满足 Hamilton 正则方程（13），所以有

$$\left[\exp\left(\int_{t_1}^{t} \frac{\partial H}{\partial z}\mathrm{d}\theta\right)\left(-\dot{p}_s - \frac{\partial H}{\partial z} - p_s\frac{\partial H}{\partial z}\right) + \varepsilon Q_s\right](\xi_s - \dot{q}_s\tau) + \exp\left(\int_{t_1}^{t} \frac{\partial H}{\partial z}\mathrm{d}\theta\right)\left(\dot{q}_s - \frac{\partial H}{\partial p_s}\right)(\eta_s - \dot{p}_s\tau) = 0 \tag{21}$$

由于

$$\frac{\mathrm{d}I_1}{\mathrm{d}t} = \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial q_s}\dot{q}_s + \frac{\partial F_0}{\partial p_s}\dot{p}_s + \frac{\partial F_0}{\partial z}\dot{z} + \varepsilon\left(\frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial q_s}\dot{q}_s + \frac{\partial F_1}{\partial p_s}\dot{p}_s + \frac{\partial F_1}{\partial z}\dot{z}\right) \tag{22}$$

根据（19）式，并综合（21），（22）两式则可得

$$\begin{aligned}&\frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial q_s}\dot{q}_s + \frac{\partial F_0}{\partial p_s}\dot{p}_s + \frac{\partial F_0}{\partial z}\dot{z} + \varepsilon\left(\frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial q_s}\dot{q}_s + \frac{\partial F_1}{\partial p_s}\dot{p}_s + \frac{\partial F_1}{\partial z}\dot{z}\right) \\ &\quad + \left\{\left[\exp\left(\int_{t_1}^{t} \frac{\partial H}{\partial z}\mathrm{d}\theta\right)\left(-\dot{p}_s - \frac{\partial H}{\partial z} - p_s\frac{\partial H}{\partial z}\right) + \varepsilon Q_s\right](\xi_s - \dot{q}_s\tau)\right. \\ &\quad \left. + \exp\left(\int_{t_1}^{t} \frac{\partial H}{\partial z}\mathrm{d}\theta\right)\left(\dot{q}_s - \frac{\partial H}{\partial p_s}\right)(\eta_s - \dot{p}_s\tau)\right\} \\ &= \varepsilon^2 Q_s(\xi_s^1 - \dot{q}_s\tau^1)\end{aligned} \tag{23}$$

其中，$\tau = \tau^0 + \varepsilon\tau^1 + \varepsilon^2\tau^2 + \cdots$，$\xi_s = \xi_s^0 + \varepsilon\xi_s^1 + \varepsilon^2\xi_s^2 + \cdots$。将（23）式进行整理后得

$$\begin{aligned}&\frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial q_s}\dot{q}_s + \frac{\partial F_0}{\partial p_s}\dot{p}_s + \frac{\partial F_0}{\partial z}\dot{z} + \exp\left(\int_{t_1}^{t} \frac{\partial H}{\partial z}\mathrm{d}\theta\right)(-\dot{p}_s)(\xi_s^0 - \dot{q}_s\tau^0) \\ &\quad + \left[\exp\left(\int_{t_1}^{t} \frac{\partial H}{\partial z}\mathrm{d}\theta\right)\left(-\frac{\partial H}{\partial q_s} - p_s\frac{\partial H}{\partial z}\right) + \varepsilon Q_s\right](\xi_s^0 - \dot{q}_s\tau^0) \\ &\quad + \exp\left(\int_{t_1}^{t} \frac{\partial H}{\partial z}\mathrm{d}\theta\right)\left(\dot{q}_s - \frac{\partial H}{\partial p_s}\right)(\eta_s - \dot{p}_s\tau) + \varepsilon\left\{\frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial q_s}\dot{q}_s + \frac{\partial F_1}{\partial p_s}\dot{p}_s + \frac{\partial F_1}{\partial z}\dot{z}\right. \\ &\quad \left. + \exp\left(\int_{t_1}^{t} \frac{\partial H}{\partial z}\mathrm{d}\theta\right)\left(-\dot{p}_s - \frac{\partial H}{\partial q_s} - p_s\frac{\partial H}{\partial q_s}\right)(\dot{\xi}_s^1 - \dot{q}_s\tau^1)\right\} = 0\end{aligned} \tag{24}$$

因此，若有

$$\frac{\partial F_0}{\partial p_s}-\exp\left(\int_{t_1}^{t}\frac{\partial H}{\partial z}\mathrm{d}\theta\right)(\xi_s^0-\dot{q}_s\tau^0)=0 \tag{25}$$

即

$$\xi_s^0-\dot{q}_s\tau^0=\exp\left(-\int_{t_1}^{t}\frac{\partial H}{\partial z}\mathrm{d}\theta\right)\frac{\partial F_0}{\partial p_s} \tag{26}$$

进一步假设

$$F_0=\exp\left(\int_{t_1}^{t}\frac{\partial H}{\partial z}\mathrm{d}\theta\right)(p_s\xi_s^0-H\tau^0+G^0) \tag{27}$$

由（26）和（27）式可以解得无扰动部分对应的生成函数分别为

$$\tau^0=(p_s\dot{q}_s-H)^{-1}\left[\exp\left(-\int_{t_1}^{t}\frac{\partial H}{\partial z}\mathrm{d}\theta\right)\left(F_0-\frac{\partial F_0}{\partial p_s}p_s\right)-G^0\right] \tag{28}$$

$$\xi_s^0=\exp\left(-\int_{t_1}^{t}\frac{\partial H}{\partial z}\mathrm{d}\theta\right)\frac{\partial F_0}{\partial p_s}+\dot{q}_s(p_s\dot{q}_s-H)^{-1}\left[\exp\left(-\int_{t_1}^{t}\frac{\partial H}{\partial z}\mathrm{d}\theta\right)\left(F_0-\frac{\partial F_0}{\partial p_s}p_s\right)-G^0\right] \tag{29}$$

进一步地分析，可以得到小扰动作用下生成函数的摄动项的结果为

$$\tau^1=(p_s\dot{q}_s-H)^{-1}\left[\exp\left(-\int_{t_1}^{t}\frac{\partial H}{\partial z}\mathrm{d}\theta\right)\left(F_1-\frac{\partial F_1}{\partial p_s}p_s\right)-G^1\right] \tag{30}$$

$$\xi_s^1=\exp\left(-\int_{t_1}^{t}\frac{\partial H}{\partial z}\mathrm{d}\theta\right)\frac{\partial F_1}{\partial p_s}+\dot{q}_s(p_s\dot{q}_s-H)^{-1}\left[\exp\left(-\int_{t_1}^{t}\frac{\partial H}{\partial z}\mathrm{d}\theta\right)\left(F_1-\frac{\partial F_1}{\partial p_s}p_s\right)-G^1\right] \tag{31}$$

于是有

定理 3　如果相空间中非保守系统（2）在小扰动 εQ_s 作用下存在一个一阶绝热不变量，形如（20）式，则存在相应的无穷小变换，无摄动项的生成函数为（28）和（29），摄动项的生成函数为（30）和（31）。

如果令

$$H(t,q_s,p_s,z)=p_s\dot{q}_s-L(t,q_s,\dot{q}_s,z) \tag{32}$$

$$p_s=\frac{\partial L}{\partial \dot{q}_s}(s=1,2,\cdots,n) \tag{33}$$

无穷小变换（5）可写成

$$\begin{aligned} t^* &= t+\varepsilon\tau^0(t,q_k,\dot{q}_k)q_s^*(t^*) \\ &= q_s(t)+\varepsilon\xi_s^0(t,q_k,\dot{q}_k)(s=1,2,\cdots,n) \end{aligned} \tag{34}$$

于是定理 2 和定理 3 给出位形空间中 Herglotz 变分问题的绝热不变量。我们有如下推论：

推论 1　如果位形空间中非保守系统受到小扰动 εQ_s 的作用，若存在规范函数 G^j，使无穷小变换的生成函数 τ^j 和 ξ_s^j 满足

$$\begin{aligned} &\exp\left(-\int_{t_1}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left[\frac{\partial L}{\partial t}\tau^j+\frac{\partial L}{\partial q_s}\xi_s^j+\frac{\partial L}{\partial \dot{q}_s}\dot{\xi}_s^j+\left(L-\frac{\partial L}{\partial \dot{q}_s}\dot{q}_s\right)\dot{\tau}^j-\frac{\partial L}{\partial z}G^j+\dot{G}^j\right] \\ &= Q_s(\xi_s^{j-1}-\dot{q}_s\tau^{j-1})(j=0,1,2,\cdots) \end{aligned} \tag{35}$$

则

$$I_m=\sum_{j=0}^{m}\varepsilon^j\exp\left(-\int_{t_1}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(\frac{\partial L}{\partial \dot{q}_s}\xi_s^j+\left(L-\frac{\partial L}{\partial \dot{q}_s}\dot{q}_s\right)\tau^j+G^j\right) \tag{36}$$

是该系统的一个 m 阶绝热不变量。

推论 2　如果位形空间中非保守系统在小扰动 εQ_s 作用下存在一个一阶绝热不变量，形如

$$I_1=F_0(t,q_s,\dot{q}_s,z)+\varepsilon F_1(t,q_s,\dot{q}_s,z) \tag{37}$$

则存在相应的无穷小变换，得到无摄动项的生成函数为（38）和（39）和摄动项的生成函数为（40）和（41）。

$$\tau^0 = L^{-1}\left[\exp\left(\int_{t_1}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(F_0 - \frac{\partial L}{\partial \dot q_l}h_{sl}\frac{\partial F_0}{\partial \dot q_s}\right) - G^0\right] \tag{38}$$

$$\xi_l^0 = h_{sl}\exp\left(\int_{t_1}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\frac{\partial F_0}{\partial \dot q_s} + \dot q_l L^{-1}\left[\exp\left(\int_{t_1}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(F_0 - \frac{\partial L}{\partial \dot q_l}h_{sl}\frac{\partial F_0}{\partial \dot q_s}\right) - G^0\right] \tag{39}$$

$$\tau^1 = L^{-1}\left[\exp\left(\int_{t_1}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(F_1 - \frac{\partial L}{\partial \dot q_l}h_{sl}\frac{\partial F_1}{\partial \dot q_s}\right) - G^1\right] \tag{40}$$

$$\xi_l^1 = h_{sl}\exp\left(\int_{t_1}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\frac{\partial F_1}{\partial \dot q_s} + \dot q_l L^{-1}\left[\exp\left(\int_{t_1}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(F_1 - \frac{\partial L}{\partial \dot q_l}h_{sl}\frac{\partial F_1}{\partial \dot q_s}\right) - G^1\right] \tag{41}$$

其中，$h_{sl} = (H_{sl})^{-1}$，$\frac{\partial^2 L}{\partial \dot q_s \partial \dot q_l} = H_{sl}$，$H_{sl}$ 称为 Hessian 矩阵。

4 算 例

研究平方阻尼振子，其运动微分方程为[28]

$$\ddot q + \gamma \dot q^2 + q = 0 \tag{42}$$

Herglotz 型 Hamilton 正则方程给出[21]

$$\dot q = p + 2\gamma z, \dot p = -2p\gamma(p + 2\gamma) - \frac{1}{2\gamma} \tag{43}$$

设无穷小生成函数 τ^0、ξ_s^0 和规范函数 G^0 满足条件（11），即

$$-\frac{1}{2\gamma}\xi^0 + p\dot\xi^0 - \left[\frac{1}{2}(p+2\gamma)^2 + \frac{q}{2\gamma} - \frac{1}{4\gamma^2}\right]\dot\tau^0 + \dot q\eta^0 - (p+2\gamma)\eta^0 = -[2\gamma(p+2\gamma)]G^0 - \dot G^0 \tag{44}$$

方程（44）有解

$$\tau^0 = -1, \xi^0 = 0, G^0 = \exp\left[-2\gamma\int_{t_1}^{t}(p+2\gamma)\mathrm{d}\theta\right], \eta^0 = a \tag{45}$$

由定理 1，该系统的一个精确不变量为

$$I_0 = \exp\left[2\gamma\int_{t_1}^{t}(p+2\gamma z)\mathrm{d}\theta\right]\left[\frac{1}{2}(p+2\gamma z)^2 + \frac{q}{2\gamma} - \frac{1}{4\gamma^2}\right] + 1 = \mathrm{const.} \tag{46}$$

下面研究系统的绝热不变量。假设系统受到的小扰动为

$$\varepsilon Q = 2\gamma\varepsilon\exp\left[2\gamma\int_{t_1}^{t}(p+2\gamma)\mathrm{d}\theta\right] \tag{47}$$

方程（17）给出

$$\begin{aligned}&\exp\left[2\gamma\int_{t_1}^{t}(p+2\gamma)\mathrm{d}\theta\right]\Big\{-\frac{1}{2\gamma}\xi^1 + p\dot\xi^1 - \left[\frac{1}{2}(p+2\gamma)^2 + \frac{q}{2\gamma} - \frac{1}{4\gamma^2}\right]\dot\tau^1 \\ &\quad + \dot q\eta^1 - (p+2\gamma)\eta^1 + [2\gamma(p+2\gamma)]G^1 + \dot G^1\Big\} \\ &= 2\gamma\exp\left[2\gamma\int_{t_1}^{t}(p+2\gamma)\mathrm{d}\theta\right](\xi^0 - \dot q\tau^0)\end{aligned} \tag{48}$$

方程（48）有解

$$\tau^1 = -1, \xi^1 = 0, G^1 = 1, \eta^1 = a \tag{49}$$

由定理 2，则该系统有如下一阶绝热不变量

$$\begin{aligned}I_1 &= \exp\left[2\gamma\int_{t_1}^{t}(p+2\gamma)\mathrm{d}\theta\right]\left\{\frac{1}{2}(p+2\gamma)^2 + \frac{q}{2\gamma} - \frac{1}{4\gamma^2} + \exp\left[-2\gamma\int_{t_1}^{t}(p+2\gamma)\mathrm{d}\theta\right]\right\} \\ &\quad + \varepsilon\left\{\exp\left[2\gamma\int_{t_1}^{t}(p+2\gamma)\mathrm{d}\theta\right]\left[\frac{1}{2}(p+2\gamma)^2 + \frac{q}{2\gamma} - \frac{1}{4\gamma^2} + 1\right]\right\}\end{aligned} \tag{50}$$

类似地，可求得系统的更高阶绝热不变量。

最后研究逆问题。假设系统受到小扰动（47）的作用，且存在一阶绝热不变量（50），则由（26）和（27）式得到

$$\xi^0 - \dot{q}\tau^0 = \dot{q} \tag{51}$$

$$\exp\left[2\gamma\int_{t_1}^{t}(p+2\gamma)\mathrm{d}\theta\right]\left\{\frac{1}{2}(p+2\gamma)^2+\frac{q}{2\gamma}-\frac{1}{4\gamma^2}+\exp\left[-2\gamma\int_{t_1}^{t}(p+2\gamma)\mathrm{d}\theta\right]\right\}$$
$$=\exp\left[2\gamma\int_{t_1}^{t}(p+2\gamma)\mathrm{d}\theta\right]\left\{p\xi^0-\left[\frac{1}{2}(p+2\gamma)^2+\frac{q}{2\gamma}-\frac{1}{4\gamma^2}\right]\tau^0+G^0\right\} \tag{52}$$

若取

$$G^0 = \exp\left[-2\gamma\int_{t_1}^{t}(p+2\gamma)\mathrm{d}\theta\right] \tag{53}$$

则有

$$\tau^0 = -1, \xi^0 = 0 \tag{54}$$

若取

$$G^1 = 1 \tag{55}$$

由式（30）和（31）可得到

$$\tau^1 = -1, \xi^1 = 0 \tag{56}$$

因此，生成元（54）和（56）给出与一阶绝热不变量（50）相应的无穷小变换。

5　结　论

文章基于相空间中 Herglotz 型微分变分原理，导出了一类新型绝热不变量。主要工作包括：一是基于 Herglotz 型微分变分原理给出了相空间中非保守系统的精确不变量（12）及其存在条件（11）；二是给出了相空间中非保守系统的一类新型绝热不变量（18），并加以证明；三是讨论了绝热不变量的逆问题。主要结果为三个定理及其两个推论。本文的方法和结果可进一步加以推广和应用，如基于 Herglotz 型微分变分原理构建 Birkhoff 系统或非完整系统的绝热不变量等。

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