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一类具有潜伏感染细胞的时滞病毒感染模型*

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摘　要：提出了一类具有潜伏感染细胞的时滞病毒感染模型，定义了基本再生数，给出了每个平衡点存在的充分条件。通过构造Lyapunov函数和利用LaSalle不变集原理，证明了当基本再生数小于或等于1时，无病平衡点是全局渐近稳定的；当基本再生数大于1时，慢性感染平衡点是全局渐近稳定的，但无病平衡点是不稳定的。结论表明，模型中的潜伏感染时滞、内时滞和病毒产生时滞并不影响模型的全局稳定性，并通过数值模拟验证了所得理论结果。

关键词：潜伏感染细胞；Beddington-DeAngelis发生率；时滞；病毒感染模型；全局稳定性

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A delayed virus infection model with latent infection cells

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Abstract：A class of delayed virus infection models with latently infected cells are investigated. The basic reproduction number is defined，and the sufficient conditions for the existence of each feasible equilibrium are given. By using Lyapunov functionals and LaSalle′s invariance principle，it is proved that when the basic reproduction number is less than or equal to unity，the infection-free equilibrium is globally asymptotically stable；when the basic reproduction number is greater than unity，the chronic-infection equilibrium is globally asymptotically stable，but the infection-free equilibrium is unstable. The results show that the latently infected delay，the intracellular delay，and virus production period in the model do not affect the global stability of the model，and numerical simulations are carried out to illustrate the theoretical results.

Key words：latently infected cells；Beddington-DeAngelis incidence；delay；virus infection model；global stability

传染病是由各种病原体引起的能在人与人、动物与动物或者人与动物之间相互传播的一类疾病[1]。大多数传染病是通过病毒感染来实现的，因此运用数学建模的方法对传染病病毒感染的动力学行为进行研究，对传染病的预防和控制均具有重要意义[2-5]。最初的病毒感染模型是由Nowak和Perelson等提出[6-8]，形式如下：

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$$\begin{cases} x'(t) = \lambda - dx(t) - \beta x(t)v(t), \\ y'(t) = \beta x(t)v(t) - ay(t), \\ v'(t) = ky(t) - uv(t) \end{cases} \tag{1}$$

其中$x(t), y(t), v(t)$分别表示t时刻未感染细胞数、被感染细胞数和病毒数。参数λ表示未感染细胞的固有生成率，β表示病毒感染率，d, a, u分别表示未感染细胞、被感染细胞和病毒的死亡率，k示病毒复制率，$\lambda, \beta, d, a, u, k$均为正数。

在模型（1）中，假定t时刻未感染细胞数和病毒数之间的发生率是双线性的，然而实际发生率可能并不是完全线性的[9-11]。另外以上提到的模型忽略了一个事实，即在细胞中并不是所有的病毒都能启动活性病毒的产生。也就是说，一部分细胞在被病毒激活感染后，进入染病阶段，但还有一部分细胞在被激活之后长时间保持静止，仍然保留在潜伏期[12-13]，这种保留在潜伏期的细胞被定义为潜伏感染细胞。这种持续潜伏在细胞内的特性目前被认为是细胞从感染中恢复的障碍，但目前，在病毒感染模型中考虑潜伏感染细胞影响的模型并不多见[14-17]，其中Wang[17]讨论了一类具有潜伏感染细胞和饱和发生率的HIV-1传染病模型：

$$\begin{cases} x'(t) = \lambda - dx(t) - \dfrac{\beta x(t)v(t)}{1 + \alpha v(t)}, \\ \omega'(t) = \dfrac{(1-q)\beta x(t)v(t)}{1 + \alpha v(t)} - e\omega(t) - \delta\omega(t), \\ y'(t) = \dfrac{q\beta x(t)v(t)}{1 + \alpha v(t)} - ay(t) + \delta\omega(t), \\ v'(t) = ky(t) - uv(t) \end{cases} \tag{2}$$

其中$\omega(t)$表示t时刻的潜伏感染细胞数量。在该模型中，假设未感染细胞被病毒激活后，以速率$\dfrac{q\beta x(t)v(t)}{1 + \alpha v(t)}$产生感染细胞，而以速率$\dfrac{(1-q)\beta x(t)v(t)}{1 + \alpha v(t)}$保持潜伏，其中$0 < q < 1$，$\alpha > 0$。$e$表示潜伏感染细胞的死亡率，$\delta$表示潜伏感染细胞转化为感染细胞的速率，$e, \delta$均为正数。

本文考虑了一类具有潜伏感染细胞和Beddington-DeAngelis发生率的时滞病毒感染模型：

$$\begin{cases} x'(t) = \lambda - dx(t) - \dfrac{\beta x(t)v(t)}{1 + \alpha_1 x(t) + \alpha_2 v(t)}, \\ \omega'(t) = \dfrac{(1-q)\beta x(t-\tau_1)v(t-\tau_1)}{1 + \alpha_1 x(t-\tau_1) + \alpha_2 v(t-\tau_1)} - e\omega(t) - \delta\omega(t), \\ y'(t) = \dfrac{q\beta x(t-\tau_2)v(t-\tau_2)}{1 + \alpha_1 x(t-\tau_2) + \alpha_2 v(t-\tau_2)} - ay(t) + \delta\omega(t), \\ v'(t) = ky(t-\tau_3) - uv(t) \end{cases} \tag{3}$$

其中时滞τ_1表示CD4+T细胞与病毒接触并成为潜伏感染细胞所需要的时间，τ_2表示CD4+T细胞与病毒接触并被感染所需要的时间，τ_3表示病毒产生时滞，$\alpha_1 > 0, \alpha_2 > 0$，其它参数的生物学意义同上。

系统（3）满足初始条件

$$\begin{cases} x(\theta) = \phi_1(\theta),\ \omega(\theta) = \phi_2(\theta),\ y(\theta) = \phi_3(\theta),\ v(\theta) = \phi_4(\theta), \\ \phi_i(\theta) \geqslant 0,\ \theta \in [-\tau, 0],\ \phi_i(0) > 0\ (i = 1,2,3,4) \end{cases} \tag{4}$$

其中$\tau = \max\{\tau_1, \tau_2, \tau_3\}$，$(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta), \phi_4(\theta)) \in C([-\tau, 0], \mathbf{R}_+^4)$，$\mathbf{R}_+^4 = \{(x_1, x_2, x_3, x_4) | x_i \geqslant 0,\ i = 1, 2, 3, 4\}$，$C([-\tau, 0], \mathbf{R}_+^4)$表示从区间$[-\tau, 0]$到$\mathbf{R}_+^4$且具有上确界范数的Banach空间的连续泛函。

由泛函微分方程的基本理论知识可知[18]，系统（3）存在唯一满足初始条件（4）的解$(x(\theta), \omega(\theta), y(\theta), v(\theta))$，且对任意$t \geqslant 0$，都有$x(t) > 0, \omega(t) > 0, y(t) > 0, v(t) > 0$。

1 解的有界性和平衡点的存在性

1.1 解的有界性

定理 1 设$(x(t),\omega(t),y(t),v(t))$是模型（3）满足初始条件（4）的解，则解$x(t),\omega(t),y(t)$和$v(t)$是最终有界的。

证明 定义函数

$$N_1(t)=x(t)+\frac{1}{1-q}\omega(t+\tau_1)$$

对函数$N_1(t)$沿系统（3）求导数，得

$$N_1'(t)=\lambda-dx(t)-\frac{e+\delta}{1-q}\omega(t+\tau_1)\leqslant\lambda-mN_1(t)$$

其中$m=\min\{d,e+\delta\}$，因此$\limsup\limits_{t\to+\infty}N_1(t)\leqslant\dfrac{\lambda}{m}$。这表明$N_1(t)$是最终有界的，从而$x(t),\omega(t)$是最终有界的。

用同样的方法，定义函数

$$N_2(t)=x(t)+\frac{1}{q}y(t+\tau_2)$$

对函数$N_2(t)$沿系统（3）求导数，得

$$N_2'(t)=\lambda-dx(t)-\frac{a}{q}y(t+\tau_2)+\frac{\delta}{q}\omega(t+\tau_2)\leqslant\lambda+\frac{\delta\lambda}{qm}-nN_2(t)$$

其中$n=\min\{d,a\}$，因此$\limsup\limits_{t\to+\infty}N_2(t)\leqslant\dfrac{(qm+\delta)\lambda}{qmn}$。于是说明$N_2(t)$最终有界，从而$y(t)$最终有界。最后，我们从系统（3）的第四个方程很容易得出$v(t)$是最终有界的。

综上可知，解$x(t),\omega(t),y(t)$和$v(t)$是最终有界的。

1.2 平衡点的存在性

显然，系统（3）总存在无病平衡点$E_0(x_0,0,0,0)$，其中$\lambda_0=\dfrac{\lambda}{d}$。定义基本再生数

$$R_0=\frac{\beta\lambda k(eq+\delta)}{au(e+\delta)(d+\lambda\alpha_1)}$$

当$R_0>1$，意味着$\beta k(eq+\delta)>au\alpha_1(e+\delta)$，$k(\beta+d\alpha_2)(eq+\delta)>au\alpha_1(e+\delta)$。若$R_0>1$，系统（3）存在一个慢性感染平衡点$E^*(x^*,\omega^*,y^*,v^*)$，其中

$$x^*=\frac{au(e+\delta)(1+\alpha_2v^*)}{\beta k(eq+\delta)-au\alpha_1(e+\delta)},\quad \omega^*=\frac{(1-q)\beta x^*v^*}{(e+\delta)(1+\alpha_1x^*+\alpha_2v^*)}$$

$$y^*=\frac{uv^*}{k},\quad v^*=\frac{k(d+\lambda\alpha_1)(eq+\delta)(R_0-1)}{k(\beta+d\alpha_2)(eq+\delta)-au\alpha_1(e+\delta)}$$

2 平衡点的全局稳定性

2.1 无病平衡点$E_0(x_0,0,0,0)$的全局渐近稳定性

定理 2

(i) 当$R_0\leqslant1$，对任意$\tau_1\geqslant0,\tau_2\geqslant0,\tau_3\geqslant0$，系统（3）的无病平衡点$E_0(x_0,0,0,0)$是全局渐近稳定的；

(ii) 当$R_0>1$，$E_0(x_0,0,0,0)$是不稳定的。

证明 (i) 构造一个 Lyapunov 函数

$$V_1(t)=x(t)-x_0-\frac{x_0}{1+\alpha_1x_0}\int_{x_0}^{x(t)}\frac{1+\alpha_1\theta}{\theta}\mathrm{d}\theta+\frac{\delta}{eq+\delta}\omega(t)+\frac{e+\delta}{eq+\delta}y(t)$$
$$+\frac{a(e+\delta)}{k(eq+\delta)}v(t)+\frac{\delta(1-q)}{eq+\delta}\int_{t-\tau_1}^{t}\frac{\beta x(\theta)v(\theta)}{1+\alpha_1x(\theta)+\alpha_2v(\theta)}\mathrm{d}\theta$$

$$+\frac{(e+\delta)q}{eq+\delta}\int_{t-\tau_2}^{t}\frac{\beta x(\theta)v(\theta)}{1+\alpha_1x(\theta)+\alpha_2v(\theta)}\mathrm{d}\theta+\frac{a(e+\delta)}{eq+\delta}\int_{t-\tau_3}^{t}y(\theta)\mathrm{d}\theta$$

计算函数$V_1(t)$沿系统（3）的全导数，得

$$\begin{aligned}
V_1'(t)=&\left(1-\frac{x_0(1+\alpha_1x(t))}{x(t)(1+\alpha_1x_0)}\right)\left(-d(x(t)-x_0)-\frac{\beta x(t)v(t)}{1+\alpha_1x(t)+\alpha_2v(t)}\right)\\
&+\frac{\delta}{eq+\delta}\left(\frac{(1-q)\beta x(t-\tau_1)v(t-\tau_1)}{1+\alpha_1x(t-\tau_1)+\alpha_2v(t-\tau_1)}-e\omega(t)-\delta\omega(t)\right)\\
&+\frac{e+\delta}{eq+\delta}\left(\frac{q\beta x(t-\tau_2)v(t-\tau_2)}{1+\alpha_1x(t-\tau_2)+\alpha_2v(t-\tau_2)}-ay(t)+\delta\omega(t)\right)\\
&+\frac{\delta(1-q)}{eq+\delta}\left(\frac{\beta x(t)v(t)}{1+\alpha_1x(t)+\alpha_2v(t)}-\frac{\beta x(t-\tau_1)v(t-\tau_1)}{1+\alpha_1x(t-\tau_1)+\alpha_2v(t-\tau_1)}\right)\\
&+\frac{(e+\delta)q}{eq+\delta}\left(\frac{\beta x(t)v(t)}{1+\alpha_1x(t)+\alpha_2v(t)}-\frac{\beta x(t-\tau_2)v(t-\tau_2)}{1+\alpha_1x(t-\tau_2)+\alpha_2v(t-\tau_2)}\right)\\
&+\frac{a(e+\delta)}{k(eq+\delta)}\left[ky(t-\tau_3)-uv(t)\right]+\frac{a(e+\delta)}{eq+\delta}\left[y(t)-y(t-\tau_3)\right]\\
=&-\frac{d(x(t)-x_0)^2}{x(t)(1+\alpha_1x_0)}+\frac{\beta x_0v(t)(1+\alpha_1x(t))}{(1+\alpha_1x_0)(1+\alpha_1x(t)+\alpha_2v(t))}-\frac{au(e+\delta)}{k(eq+\delta)}v(t)\\
\leqslant&-\frac{d(x(t)-x_0)^2}{x(t)(1+\alpha_1x_0)}+\frac{\beta x_0v(t)}{1+\alpha_1x_0}-\frac{au(e+\delta)}{k(eq+\delta)}v(t)\\
=&-\frac{d(x(t)-x_0)^2}{x(t)(1+\alpha_1x_0)}+\frac{au(e+\delta)}{k(eq+\delta)}(R_0-1)v(t)
\end{aligned}$$

若$R_0\leqslant 1$，$V_1'(t)\leqslant 0$，当且仅当$x(t)=x_0$，$v(t)=0$时，$V_1'(t)=0$。并通过系统（3）的第二个方程和第三个方程可得，$\omega(t)=0$和$y(t)=0$。最后由LaSalle不变集原理[18]得，当$R_0\leqslant 1$时，对任意$\tau_1\geqslant 0,\tau_2\geqslant 0,\tau_3\geqslant 0$，无病平衡点$E_0$是全局渐近稳定的。

（ii）模型（3）的线性化近似系统在平衡点$E_0(x_0,0,0,0)$处的特征方程为

$$(s+d)f(s)=0 \tag{5}$$

其中

$$f(s)=\begin{vmatrix} s+e+\delta & 0 & -\dfrac{(1-q)\beta x_0e^{-s\tau_1}}{1+\alpha_1x_0}\\ -\delta & s+a & -\dfrac{q\beta x_0e^{-s\tau_2}}{1+\alpha_1x_0}\\ 0 & -ke^{-s\tau_3} & s+u\end{vmatrix}$$

显然，方程（5）总有一个负实根$s_1=-d$，因此，平衡点$E_0(x_0,0,0,0)$的稳定性取决于方程$f(s)=0$的根。通过计算

$$f(s)=(s+e+\delta)(s+a)(s+u)-\frac{\delta k(1-q)\beta x_0e^{-s(\tau_1+\tau_3)}}{1+\alpha_1x_0}-\frac{kq\beta x_0e^{-s(\tau_2+\tau_3)}}{1+\alpha_1x_0}(s+e+\delta) \tag{6}$$

若$R_0>1$，则有

$$f(0)=au(e+\delta)-\frac{k\beta x_0}{1+\alpha_1x_0}(eq+\delta)=au(e+\delta)(1-R_0)<0,\ \lim_{s\to+\infty}f(s)=+\infty$$

因此，方程（6）至少有一个正实根，从而说明当$R_0>1$时，平衡点$E_0(x_0,0,0,0)$是不稳定的。

2.2　慢性感染平衡点$E^*(x^*,\omega^*,y^*,v^*)$的全局渐近稳定性

在下面的定理中，为了表示方便，定义如下函数：

$$F(z)=z-1-\ln z$$

显然，对任意$z \in (0, +\infty)$，$F(z)$是非负的，且在$z = 1$时取到最小值$F(1) = 0$。

定理3　当$R_0 > 1$时，对任意$\tau_1 \geqslant 0, \tau_2 \geqslant 0, \tau_3 \geqslant 0$，系统（3）的慢性感染平衡点$E^*(x^*, \omega^*, y^*, v^*)$是全局渐近稳定的。

证明　构造Lyapunov函数

$$V_2(t) = x(t) - x^* - \frac{\beta x^* v^*}{1 + \alpha_1 x^* + \alpha_2 v^*}\int_{x^*}^{x(t)} \frac{1 + \alpha_1\theta + \alpha_2 v^*}{\beta\theta v^*}\mathrm{d}\theta + \frac{\delta}{eq + \delta}\omega^* F\left(\frac{\omega(t)}{\omega^*}\right)$$
$$+\frac{e + \delta}{eq + \delta}y^* F\left(\frac{y(t)}{y^*}\right) + \frac{a(e + \delta)}{k(eq + \delta)}v^* F\left(\frac{v(t)}{v^*}\right) + V_{21}(t) + V_{22}(t) + V_{23}(t)$$

其中

$$V_{21}(t) = \frac{\delta(1 - q)}{eq + \delta}\frac{\beta x^* v^*}{1 + \alpha_1 x^* + \alpha_2 v^*}\int_{t - \tau_1}^{t} F\left(\frac{x(\theta)v(\theta)(1 + \alpha_1 x^* + \alpha_2 v^*)}{x^* v^*(1 + \alpha_1 x(\theta) + \alpha_2 v(\theta))}\right)\mathrm{d}\theta,$$

$$V_{22}(t) = \frac{(e + \delta)q}{eq + \delta}\frac{\beta x^* v^*}{1 + \alpha_1 x^* + \alpha_2 v^*}\int_{t - \tau_2}^{t} F\left(\frac{x(\theta)v(\theta)(1 + \alpha_1 x^* + \alpha_2 v^*)}{x^* v^*(1 + \alpha_1 x(\theta) + \alpha_2 v(\theta))}\right)\mathrm{d}\theta,$$

$$V_{23}(t) = \frac{a(e + \delta)y^*}{eq + \delta}\int_{t - \tau_3}^{t} F\left(\frac{y(\theta)}{y^*}\right)\mathrm{d}\theta$$

计算函数$V_2(t)$沿系统（3）的全导数，得

$$V_2'(t) = \left(1 - \frac{\beta x^* v^*}{1 + \alpha_1 x^* + \alpha_2 v^*}\frac{1 + \alpha_1 x(t) + \alpha_2 v^*}{\beta x(t) v^*}\right)\left(\lambda - dx(t) - \frac{\beta x(t)v(t)}{1 + \alpha_1 x(t) + \alpha_2 v(t)}\right)$$
$$+\frac{\delta}{eq + \delta}\left(1 - \frac{\omega^*}{\omega(t)}\right)\left(\frac{(1 - q)\beta x(t - \tau_1)v(t - \tau_1)}{1 + \alpha_1 x(t - \tau_1) + \alpha_2 v(t - \tau_1)} - (e + \delta)\omega(t)\right)$$
$$+\frac{e + \delta}{eq + \delta}\left(1 - \frac{y^*}{y(t)}\right)\left(\frac{q\beta x(t - \tau_2)v(t - \tau_2)}{1 + \alpha_1 x(t - \tau_2) + \alpha_2 v(t - \tau_2)} - ay(t) + \delta\omega(t)\right)$$
$$+\frac{a(e + \delta)}{k(eq + \delta)}\left(1 - \frac{v^*}{v(t)}\right)\left[ky(t - \tau_3) - uv(t)\right] + V_{21}'(t) + V_{22}'(t) + V_{23}'(t)$$

其中

$$V_{21}'(t) = \frac{\delta(1 - q)}{eq + \delta}\left(\frac{\beta x(t)v(t)}{1 + \alpha_1 x(t) + \alpha_2 v(t)} - \frac{\beta x(t - \tau_1)v(t - \tau_1)}{1 + \alpha_1 x(t - \tau_1) + \alpha_2 v(t - \tau_1)}\right)$$
$$+\frac{\delta(1 - q)}{eq + \delta}\frac{\beta x^* v^*}{1 + \alpha_1 x^* + \alpha_2 v^*}\ln\frac{x(t - \tau_1)v(t - \tau_1)(1 + \alpha_1 x(t) + \alpha_2 v(t))}{x(t)v(t)(1 + \alpha_1 x(t - \tau_1) + \alpha_2 v(t - \tau_1))},$$

$$V_{22}'(t) = \frac{(e + \delta)q}{eq + \delta}\left(\frac{\beta x(t)v(t)}{1 + \alpha_1 x(t) + \alpha_2 v(t)} - \frac{\beta x(t - \tau_2)v(t - \tau_2)}{1 + \alpha_1 x(t - \tau_2) + \alpha_2 v(t - \tau_2)}\right)$$
$$+\frac{(e + \delta)q}{eq + \delta}\frac{\beta x^* v^*}{1 + \alpha_1 x^* + \alpha_2 v^*}\ln\frac{x(t - \tau_2)v(t - \tau_2)(1 + \alpha_1 x(t) + \alpha_2 v(t))}{x(t)v(t)(1 + \alpha_1 x(t - \tau_2) + \alpha_2 v(t - \tau_2))},$$

$$V_{23}'(t) = \frac{a(e + \delta)}{eq + \delta}\left[y(t) - y(t - \tau_3)\right] + \frac{a(e + \delta)y^*}{eq + \delta}\ln\frac{y(t - \tau_3)}{y(t)}$$

由于在平衡点E^*处，

$$y^* = \frac{uv^*}{k},\quad ay^* = \frac{q\beta x^* v^*}{1 + \alpha_1 x^* + \alpha_2 v^*} + \delta\omega^*,\quad (e + \delta)\omega^* = \frac{(1 - q)\beta x^* v^*}{1 + \alpha_1 x^* + \alpha_2 v^*},$$
$$\frac{(e + \delta)ay^*}{eq + \delta} = \frac{\beta x^* v^*}{1 + \alpha_1 x^* + \alpha_2 v^*},\quad \lambda = dx^* + \frac{\beta x^* v^*}{1 + \alpha_1 x^* + \alpha_2 v^*}$$

则

$$
\begin{aligned}
V_2'(t) = &-\frac{d(1+\alpha_2 v^*)\big(x(t)-x^*\big)^2}{x(t)(1+\alpha_1 x^*+\alpha_2 v^*)} - \frac{\beta x^* v^*}{1+\alpha_1 x^*+\alpha_2 v^*}\cdot\frac{x^*\big(1+\alpha_1 x(t)+\alpha_2 v^*\big)}{x(t)\big(1+\alpha_1 x^*+\alpha_2 v^*\big)}\\
&+\frac{\beta x^* v^*}{1+\alpha_1 x^*+\alpha_2 v^*}\cdot\frac{v(t)\big(1+\alpha_1 x(t)+\alpha_2 v^*\big)}{v^*\big(1+\alpha_1 x(t)+\alpha_2 v(t)\big)} + \frac{\delta(1-q)}{eq+\delta}\cdot\frac{\beta x^* v^*}{1+\alpha_1 x^*+\alpha_2 v^*}\\
&-\frac{\delta(1-q)}{eq+\delta}\cdot\frac{\beta x^* v^*}{1+\alpha_1 x^*+\alpha_2 v^*}\cdot\frac{\omega^*}{\omega(t)}\cdot\frac{x(t-\tau_1)v(t-\tau_1)\big(1+\alpha_1 x^*+\alpha_2 v^*\big)}{x^* v^*\big(1+\alpha_1 x(t-\tau_1)+\alpha_2 v(t-\tau_1)\big)}\\
&+\frac{\delta(1-q)}{eq+\delta}\frac{\beta x^* v^*}{1+\alpha_1 x^*+\alpha_2 v^*}\ln\frac{x(t-\tau_1)v(t-\tau_1)\big(1+\alpha_1 x(t)+\alpha_2 v(t)\big)}{x(t)v(t)\big(1+\alpha_1 x(t-\tau_1)+\alpha_2 v(t-\tau_1)\big)}\\
&-\frac{(e+\delta)q}{eq+\delta}\cdot\frac{\beta x^* v^*}{1+\alpha_1 x^*+\alpha_2 v^*}\cdot\frac{y^*}{y(t)}\cdot\frac{x(t-\tau_2)v(t-\tau_2)\big(1+\alpha_1 x^*+\alpha_2 v^*\big)}{x^* v^*\big(1+\alpha_1 x(t-\tau_2)+\alpha_2 v(t-\tau_2)\big)}\\
&+\frac{(e+\delta)q}{eq+\delta}\frac{\beta x^* v^*}{1+\alpha_1 x^*+\alpha_2 v^*}\ln\frac{x(t-\tau_2)v(t-\tau_2)\big(1+\alpha_1 x(t)+\alpha_2 v(t)\big)}{x(t)v(t)\big(1+\alpha_1 x(t-\tau_2)+\alpha_2 v(t-\tau_2)\big)}\\
&-\frac{\delta(1-q)}{eq+\delta}\cdot\frac{\beta x^* v^*}{1+\alpha_1 x^*+\alpha_2 v^*}\cdot\frac{y^*}{y(t)}\cdot\frac{\omega(t)}{\omega^*} - \frac{\beta x^* v^*}{1+\alpha_1 x^*+\alpha_2 v^*}\cdot\frac{v^*}{v(t)}\cdot\frac{y(t-\tau_3)}{y^*}\\
&+\frac{\beta x^* v^*}{1+\alpha_1 x^*+\alpha_2 v^*}\ln\frac{y(t-\tau_3)}{y^*} - \frac{\beta x^* v^*}{1+\alpha_1 x^*+\alpha_2 v^*}\cdot\frac{v(t)}{v^*} + \frac{3\beta x^* v^*}{1+\alpha_1 x^*+\alpha_2 v^*}\\
= &-\frac{d(1+\alpha_2 v^*)\big(x(t)-x^*\big)^2}{x(t)(1+\alpha_1 x^*+\alpha_2 v^*)} + \frac{\delta(1-q)}{eq+\delta}\cdot\frac{\beta x^* v^*}{1+\alpha_1 x^*+\alpha_2 v^*}\cdot\\
&\left(5-\frac{x^*\big(1+\alpha_1 x(t)+\alpha_2 v^*\big)}{x(t)\big(1+\alpha_1 x^*+\alpha_2 v^*\big)} - \frac{\omega^*}{\omega(t)}\cdot\frac{x(t-\tau_1)v(t-\tau_1)\big(1+\alpha_1 x^*+\alpha_2 v^*\big)}{x^* v^*\big(1+\alpha_1 x(t-\tau_1)+\alpha_2 v(t-\tau_1)\big)}\right)\\
&+\frac{\delta(1-q)}{eq+\delta}\cdot\frac{\beta x^* v^*}{1+\alpha_1 x^*+\alpha_2 v^*}\left(-\frac{y^*\omega(t)}{y(t)\omega^*} - \frac{v^* y(t-\tau_3)}{v(t)y^*} - \frac{1+\alpha_1 x(t)+\alpha_2 v(t)}{1+\alpha_1 x(t)+\alpha_2 v^*}\right)\\
&+\frac{\delta(1-q)}{eq+\delta}\cdot\frac{\beta x^* v^*}{1+\alpha_1 x^*+\alpha_2 v^*}\left(\ln\frac{x(t-\tau_1)v(t-\tau_1)\big(1+\alpha_1 x(t)+\alpha_2 v(t)\big)}{x(t)v(t)\big(1+\alpha_1 x(t-\tau_1)+\alpha_2 v(t-\tau_1)\big)} + \ln\frac{y(t-\tau_3)}{y(t)}\right)\\
&+\frac{(e+\delta)q}{eq+\delta}\cdot\frac{\beta x^* v^*}{1+\alpha_1 x^*+\alpha_2 v^*}\cdot\left(4-\frac{x^*\big(1+\alpha_1 x(t)+\alpha_2 v^*\big)}{x(t)\big(1+\alpha_1 x^*+\alpha_2 v^*\big)} - \frac{v^* y(t-\tau_3)}{v(t)y^*}\right)\\
&+\frac{(e+\delta)q}{eq+\delta}\cdot\frac{\beta x^* v^*}{1+\alpha_1 x^*+\alpha_2 v^*}\cdot\left(-\frac{1+\alpha_1 x(t)+\alpha_2 v(t)}{1+\alpha_1 x(t)+\alpha_2 v^*} - \frac{y^*}{y(t)}\cdot\frac{x(t-\tau_2)v(t-\tau_2)\big(1+\alpha_1 x^*+\alpha_2 v^*\big)}{x^* v^*\big(1+\alpha_1 x(t-\tau_2)+\alpha_2 v(t-\tau_2)\big)}\right)\\
&+\frac{(e+\delta)q}{eq+\delta}\cdot\frac{\beta x^* v^*}{1+\alpha_1 x^*+\alpha_2 v^*}\cdot\left(\ln\frac{x(t-\tau_2)v(t-\tau_2)\big(1+\alpha_1 x(t)+\alpha_2 v(t)\big)}{x(t)v(t)\big(1+\alpha_1 x(t-\tau_2)+\alpha_2 v(t-\tau_2)\big)} + \ln\frac{y(t-\tau_3)}{y(t)}\right)\\
&+\frac{\beta x^* v^*}{1+\alpha_1 x^*+\alpha_2 v^*}\left(\frac{1+\alpha_1 x(t)+\alpha_2 v(t)}{1+\alpha_1 x(t)+\alpha_2 v^*} - 1 + \frac{v(t)\big(1+\alpha_1 x(t)+\alpha_2 v^*\big)}{v^*\big(1+\alpha_1 x(t)+\alpha_2 v(t)\big)} - \frac{v(t)}{v^*}\right)
\end{aligned}
$$

注意到

$$
\frac{1+\alpha_1 x(t)+\alpha_2 v(t)}{1+\alpha_1 x(t)+\alpha_2 v^*} - 1 + \frac{v(t)\big(1+\alpha_1 x(t)+\alpha_2 v^*\big)}{v^*\big(1+\alpha_1 x(t)+\alpha_2 v(t)\big)} - \frac{v(t)}{v^*} = -\frac{\alpha_2(1+\alpha_1 x(t))(v(t)-v^*)^2}{v^*(1+\alpha_1 x(t)+\alpha_2 v^*)\big(1+\alpha_1 x(t)+\alpha_2 v(t)\big)}
$$

从而

$$
\begin{aligned}
V_2'(t) = & -\frac{d(1+\alpha_2 v^*)\big(x(t)-x^*\big)^2}{x(t)(1+\alpha_1 x^*+\alpha_2 v^*)} - \frac{\delta(1-q)}{eq+\delta}\cdot\frac{\beta x^* v^*}{1+\alpha_1 x^*+\alpha_2 v^*}F\left(\frac{y^*\omega(t)}{y(t)\omega^*}\right) \\
& -\frac{\delta(1-q)}{eq+\delta}\cdot\frac{\beta x^* v^*}{1+\alpha_1 x^*+\alpha_2 v^*}\left[F\left(\frac{x^*\big(1+\alpha_1 x(t)+\alpha_2 v^*\big)}{x(t)\big(1+\alpha_1 x^*+\alpha_2 v^*\big)}\right)+F\left(\frac{1+\alpha_1 x(t)+\alpha_2 v(t)}{1+\alpha_1 x(t)+\alpha_2 v^*}\right)\right] \\
& -\frac{\delta(1-q)}{eq+\delta}\cdot\frac{\beta x^* v^*}{1+\alpha_1 x^*+\alpha_2 v^*}\left[F\left(\frac{\omega^* x(t-\tau_1)v(t-\tau_1)\big(1+\alpha_1 x^*+\alpha_2 v^*\big)}{x^* v^*\omega(t)\big(1+\alpha_1 x(t-\tau_1)+\alpha_2 v(t-\tau_1)\big)}\right)+F\left(\frac{v^* y(t-\tau_3)}{v(t)y^*}\right)\right] \\
& -\frac{(e+\delta)q}{eq+\delta}\cdot\frac{\beta x^* v^*}{1+\alpha_1 x^*+\alpha_2 v^*}\left[F\left(\frac{1+\alpha_1 x(t)+\alpha_2 v(t)}{1+\alpha_1 x(t)+\alpha_2 v^*}\right)+F\left(\frac{x^*(1+\alpha_1 x(t)+\alpha_2 v^*)}{x(t)(1+\alpha_1 x^*+\alpha_2 v^*)}\right)\right] \\
& -\frac{(e+\delta)q}{eq+\delta}\cdot\frac{\beta x^* v^*}{1+\alpha_1 x^*+\alpha_2 v^*}\left[F\left(\frac{y^* x(t-\tau_2)v(t-\tau_2)\big(1+\alpha_1 x^*+\alpha_2 v^*\big)}{x^* v^* y(t)\big(1+\alpha_1 x(t-\tau_2)+\alpha_2 v(t-\tau_2)\big)}\right)+F\left(\frac{v^* y(t-\tau_3)}{v(t)y^*}\right)\right] \\
& -\frac{\beta x^*\alpha_2(1+\alpha_1 x(t))(v(t)-v^*)^2}{(1+\alpha_1 x^*+\alpha_2 v^*)(1+\alpha_1 x(t)+\alpha_2 v^*)\big(1+\alpha_1 x(t)+\alpha_2 v(t)\big)}
\end{aligned}
$$

由于$\forall z>0$，$F(z)\geqslant 0$，且有$F_{\min}=F(1)=0$，所以当$R_0>1$时，$V_2'(t)\leqslant 0$，当且仅当$x(t)=x^*$，$\omega(t)=\omega^*$，$v(t)=v^*$，$y(t)=y^*$时，$V_2'(t)=0$。从而通过 LaSalle 不变集原理[18]得，当$R_0>1$时，对任意$\tau_1\geqslant 0$，$\tau_2\geqslant 0$，$\tau_3\geqslant 0$，E^*是全局渐近稳定的。

3 数值模拟

在系统（3）中，若令参数

$$\lambda=5, d=0.1, \beta=0.45, a=0.9, k=0.4, u=10, q=0.8, \alpha_1=0.01, \alpha_2=0.01, e=0.3, \delta=0.5$$

并给定初始值

$$x(0)=2,\ \omega(0)=0.5, y(0)=3, v(0)=0.3$$

则基本再生数

$$R_0=\frac{\beta\lambda k(eq+\delta)}{au(e+\delta)(d+\lambda\alpha_1)}\approx 0.6167<1$$

由定理2知，系统（3）的无病平衡点$E_0(x_0,0,0,0)$是全局渐近稳定的，其中$\lambda_0=\dfrac{\lambda}{d}=50$，此时数值模拟验证了所得结论，其在图1中(a), (b), (c), (d)分别表示$x(t),\omega(t),y(t),v(t)$随时间变化的曲线图。并可验证，时滞τ_1,τ_2,τ_3对无病平衡点$E_0(x_0,0,0,0)$的值和稳定性均无影响。

若令参数

$$\lambda=10, d=0.01, \beta=0.45, a=0.8, k=0.4, u=3, q=0.8, \alpha_1=0.01, \alpha_2=0.01, e=0.3, \delta=0.5$$

并给定初始值

$$x(0)=2,\ \omega(0)=0.5, y(0)=3, v(0)=0.3$$

则基本再生数$R_0=\dfrac{\beta\lambda k(eq+\delta)}{au(e+\delta)(d+\lambda\alpha_1)}\approx 6.3068>1$，系统（3）存在唯一的慢性感染平衡点

$$E^*(17.0973, 2.6211, 11.3648, 1.5153)$$

下面分别给出当$\tau_1=3,\tau_2=5,\tau_3=2$和$\tau_1=30,\tau_2=50,\tau_3=20$时的数值模拟，其中在图2–图3中，(a), (b), (c), (d)分别表示$x(t),\omega(t),y(t),v(t)$随时间变化的曲线图。

经过多次模拟验证，当$R_0>1$时，对任意$\tau_1\geqslant 0$，$\tau_2\geqslant 0$，$\tau_3\geqslant 0$，系统（3）的慢性感染平衡点E^*是全局渐近稳定的，与所得理论结果一致。

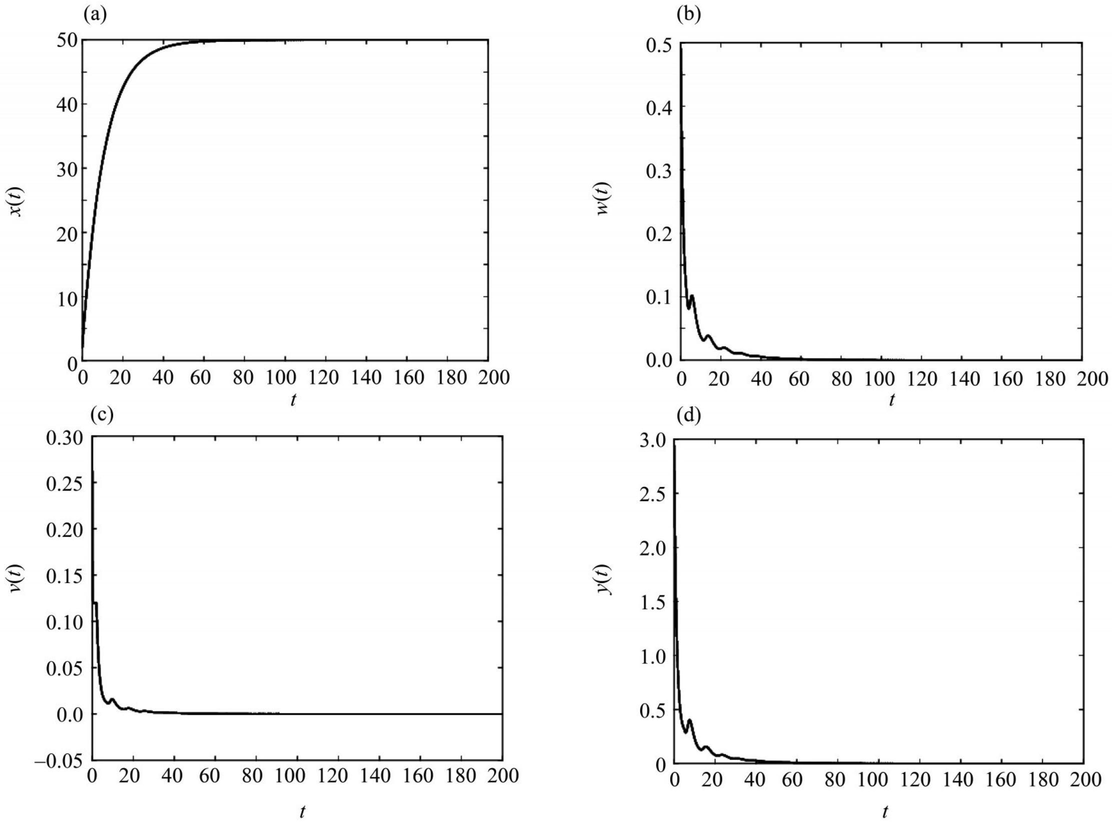

图1　当 $R_0 \leqslant 1$ 时，无病平衡点 E_0 的时间序列图

Fig. 1　When $R_0 \leqslant 1$, time series diagram of infection-free equilibrium E_0

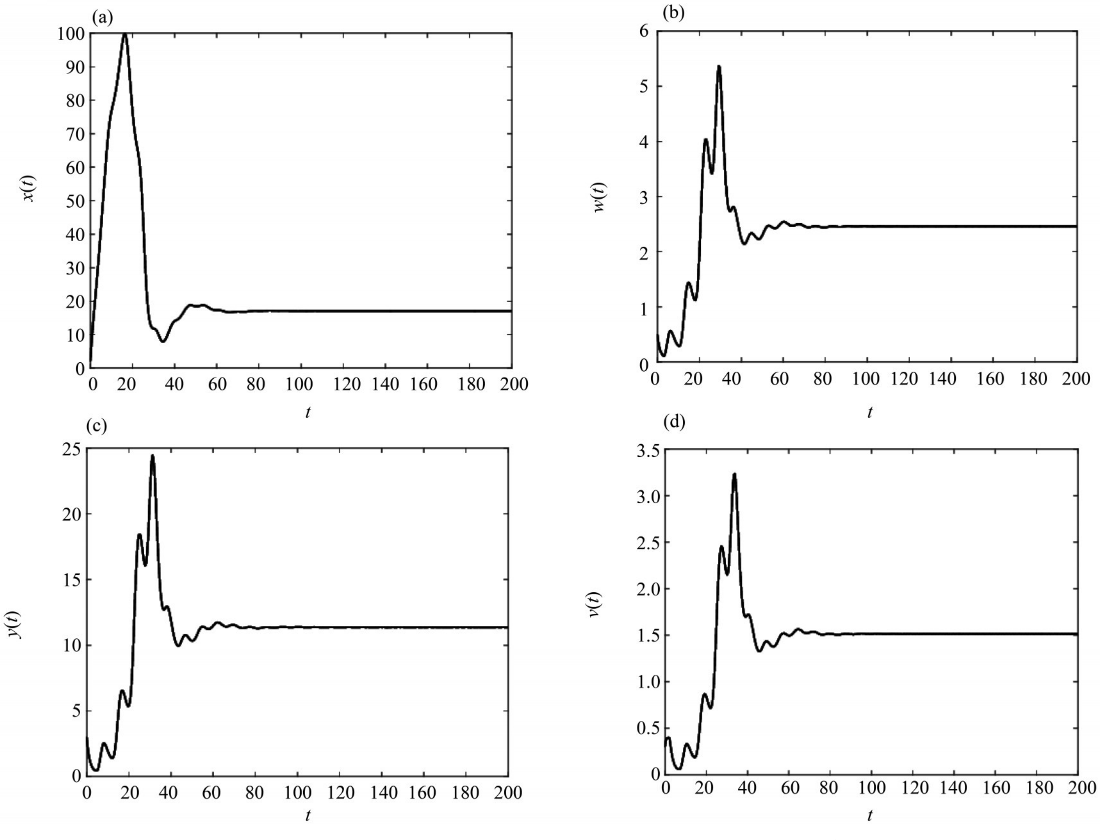

图2　当 $R_0 > 1$ 时，$\tau_1 = 3, \tau_2 = 5, \tau_3 = 2$ 时，平衡点 E^* 的时间序列图

Fig. 2　When $R_0 > 1$, $\tau_1 = 3, \tau_2 = 5, \tau_3 = 2$, time series diagram of equilibrium E^*

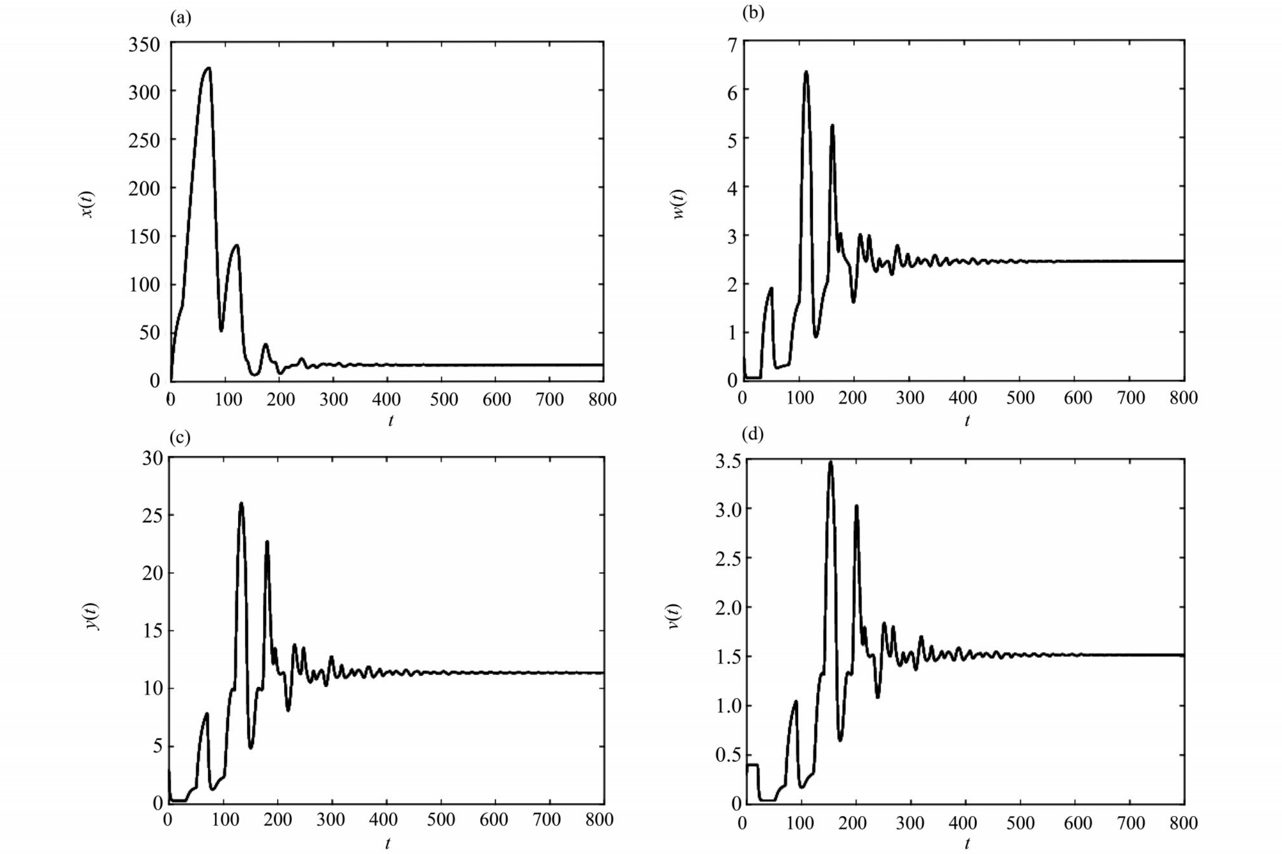

图3　当$R_0 > 1$时，$\tau_1 = 30, \tau_2 = 50, \tau_3 = 20$时，平衡点$E^*$的时间序列图

Fig. 3　When $R_0 > 1$, $\tau_1 = 30, \tau_2 = 50, \tau_3 = 20$, time series diagram of equilibrium E^*

4　结　论

本文研究了一类具有潜伏感染细胞和Beddington-DeAngelis发生率的时滞病毒感染模型，讨论了系统（3）的无病平衡点E_0和正平衡点E^*的全局渐近稳定性。结论表明：当$R_0 \leqslant 1$时，对任意$\tau_1 \geqslant 0$，$\tau_2 \geqslant 0$，$\tau_3 \geqslant 0$，系统（3）的无病平衡点E_0是全局渐近稳定的，即在这种情况下，细胞没有被传染；当$R_0 > 1$时，对任意$\tau_1 \geqslant 0$，$\tau_2 \geqslant 0$，$\tau_3 \geqslant 0$，系统（3）的慢性感染平衡点E^*是全局渐近稳定的，即CD4+T细胞被病毒感染后，一部分被激活进入染病阶段，但另一部分在被激活之后长时间保持静止，这种持续潜伏的状态成为细胞从感染中恢复的障碍，应引起大家的重视。在后期的研究中，我们会进一步考虑加入免疫应答细胞以及免疫时滞的情形。

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