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Gorenstein homological modules and Auslander categories with respect to a semidualizing module*

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Abstract: Some new characterizations on Gorenstein injective, projective and flat modules with a semidualizing R-module C are given by Auslander categories over a commutative noetherian local ring R.

Key words: semidualizing module; Auslander category; G_C-projective(injective, flat) module

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Throughout this paper, all rings are commutative noetherian local. Assume that R has a dualizing complex D. The Auslander categories $\mathrm{A}(R)$ and $\mathrm{B}(R)$ with respect to D are defined (Avramov et al. ,1997). It was shown that the modules in $\mathrm{A}(R)$ are those of finite Gorenstein projective dimension (Gorenstein flat dimension) and the modules in $\mathrm{B}(R)$ are those of finite Gorenstein injective dimension (Christensen et al. , 2006). These give resolution-free characterizations of modules for which their Gorenstein homological dimensions are finite.

Since the dualizing complexes (or modules) for a general ring usually do not exist, then the semidualizing modules (see Definition 1) and complexes have received much attention in recent years, see (Christensen,2001; Holm et al. ,2006;Holm et al. ,2007;Christensen et al. ,2009;Sather-Wagstaff et al. ,2009;Takahashi et al. ,2010; White, 2010; Sather-Wagstaff et al. ,2011;Di et al. ,2012) for instance. The examples of semidualizing modules include the rank 1 free module and a dualizing module, when one exists.

Let C be a fixed semidualizing R-module (or complex). The Auslander categories $\mathrm{A}_C(R)$ and $\mathrm{B}_C(R)$ with respect to C are defined by Christensen (2001). If R has a dualizing complex D, then these categories give resolution-free characterizations of modules for which their Gorenstein homological dimensions with respect to a semidualizing module are finite (Holm et al. ,2006).

Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring and denote by $\hat{R}$ the $\mathfrak{m}$-adic completion of R. It is well known that $\hat{R}$ has a dualizing complex.

Wu et al. (2016) proved that an R-module M is Gorenstein projective if and only if the Matlis dual $\mathrm{Hom}_R(M, E(k)) \in B(\hat{R})$ and $\mathrm{Ext}_R^{i\geqslant 1}(M, P) = 0$ for all projective R-modules P; an R-module M is Gorenstein flat if and only if the Matlis dual $\mathrm{Hom}_R(M, E(k)) \in \mathrm{B}(\hat{R})$ and $\mathrm{Tor}^R_{i\geqslant 1}(I, M) = 0$ for all injective R-modules I.

Motivated by this, it is natural to consider Gorenstein homological modules and Auslander categories with respect to a semidualizing module.

1 Preliminaries

The notion of semidualizing modules, defined next, goes back at least to Foxby (1972) .

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Definition 1 A finitely generated R-module C is semidualizing if

(a) The natural homothety morphism $R \to \mathrm{Hom}_R(C,C)$ is an isomorphism, and

(b) $\mathrm{Ext}_R^{\geqslant 1}(C,C) = 0$.

Let C be a semidualizing R-module. We set

$$\mathrm{P}_C(R) = \text{the subcategory of modules } C \otimes_R P \text{ where } P \text{ is } R\text{-projective},$$
$$\mathrm{F}_C(R) = \text{the subcategory of modules } C \otimes_R F \text{ where } F \text{ is } R\text{-flat},$$
$$\mathrm{I}_C(R) = \text{the subcategory of modules } \mathrm{Hom}_R(C,I) \text{ where } I \text{ is } R\text{-injective}.$$

Modules in $\mathrm{P}_C(R)$, $\mathrm{F}_C(R)$ and $\mathrm{I}_C(R)$ are called C-projective, C-flat and C-injective, respectively. An R-module M is C-cotorsion if $\mathrm{Ext}_R^1(C \otimes_R F, M) = 0$ for all flat R-modules F, in the case $C = R$, we call the R-module M is cotorsion.

Definition 2 Let C be a semidualizing R-module. The Auslander class $\mathrm{A}_C(R)$ with respect to C consists of all R-modules M satisfying

(i) $\mathrm{Tor}^R_{\geqslant 1}(C, M) = 0 = \mathrm{Ext}_R^{\geqslant 1}(C, C \otimes_R M)$,

(ii) the natural evaluation homomorphism $\mu_M : M \to \mathrm{Hom}_R(C, C \otimes_R M)$ is an isomorphism.

The Bass class $\mathrm{B}_C(R)$ with respect to C consists of all R-modules N satisfying

(iii) $\mathrm{Ext}_R^{\geqslant 1}(C, N) = 0 = \mathrm{Tor}^R_{\geqslant 1}(C, \mathrm{Hom}_R(C, N))$,

(iv) the natural evaluation homomorphism $\mathrm{v}_N : C \otimes_R \mathrm{Hom}_R(C,N) \to N$ is an isomorphism.

Definition 3 Let X be a class of R-modules and M an R-module. An X-resolution of M is a complex of R-modules in X of the form

$$X = \cdots \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to 0,$$

such that $H_0(X) \cong M$ and $H_n(X) = 0$ for $n \geqslant 1$, and the following exact sequence is the augmented X-resolution of M associated to X:

$$X^+ = \cdots \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to M \to 0.$$

The X-projective dimension of M is the quantity

$$\text{X-pd}_R(M) = \inf\{\sup\{n \geqslant 0 \mid X_n \neq 0\} \mid X \text{ is an X-resolution of } M\}$$

In particular, one has $\text{X-pd}_R(0) = -\infty$. The modules of X-projective dimension 0 are the nonzero modules of X. An X-resolution X of M is proper if the augmented resolution X^+ is $\mathrm{Hom}_R(\mathrm{X}, -)$-exact.

We define (proper) X-coresolutions and X-injective dimension dually. And the X-injective dimension of M is $\text{X-id}_R(M)$.

When X is the class of projective, injective and at R-modules, we write $\mathrm{pd}_R(M)$, $\mathrm{id}_R(M)$,and $\mathrm{fd}_R(M)$ for the classical homological dimensions of M. By $\overline{\mathrm{P}}_C(R)(\overline{\mathrm{P}}(R))$, $\overline{\mathrm{I}}_C(R)(\overline{\mathrm{I}}(R))$ and $\overline{\mathrm{F}}_C(R)(\overline{\mathrm{F}}(R))$, we denote the classes of R-modules with finite C-projective (projective), C-injective (injective) and C-flat (flat) dimension, respectively.

Note that if R is a commutative noetherian ring of finite Krull dimension, the $\overline{\mathrm{P}}(R) = \overline{\mathrm{F}}(R)$, and so $\overline{\mathrm{P}}_C(R) = \overline{\mathrm{F}}_C(R)$ (see Theorem 4. 2. 8 (Xu, 1996)). We shall use these facts without comment.

The next definition is due to Holm et al. (2006).

Definition 4 (i) An R-module M is called G_C-injective if

(a) $\mathrm{Ext}_R^{\geqslant 1}(\mathrm{Hom}_R(C, I), M) = 0$ for any injective R-module I,

(b) there exist injective R-modules $I_0, I_1, \cdots$ together with an exact sequence

$$X = \cdots \to \mathrm{Hom}_R(C, I_1) \to \mathrm{Hom}_R(C, I_0) \to M \to 0$$

such that this sequence stays exact when we apply the functor $\mathrm{Hom}_R(\mathrm{Hom}_R(C, E), -)$ to it for any injective R-module E (*i. e.*, M admits a proper $\mathrm{I}_C(R)$-resolution).

(ii) An R-module M is called G_C-projective if

(c) $\mathrm{Ext}_R^{\geqslant 1}(M, C \otimes_R P) = 0$ for any projective R-module P,

(d) there exist projective R-modules $P^0, P^1, \cdots$ together with an exact sequence

$$X = 0 \to M \to C \otimes_R P^0 \to C \otimes_R P^1 \to \cdots,$$

and also, this sequence stays exact when we apply the functor $\mathrm{Hom}_R(-, C \otimes_R Q)$ to it for any projective R-module Q (*i. e.*, M admits a proper $\mathrm{P}_C(R)$ -coresolution).

(iii) An R-module M is called G_C-flat if

(e) $\mathrm{Tor}^R_{\geqslant 1}(\mathrm{Hom}_R(C, I), M) = 0$ for any injective R-module I,

(f) there exist flat R-modules $F^0, F^1, \cdots$ together with an exact sequence

$$X = 0 \to M \to C \otimes_R F^0 \to C \otimes_R F^1 \to \cdots,$$

and furthermore, this sequence stays exact when we apply the functor $\mathrm{Hom}_R(C, E) \otimes_R -$ to it for any injective R-module E. We set

$\mathrm{GI}_C(R)$ = the subcategory of G_C-injective R-modules,

$\mathrm{GP}_C(R)$ = the subcategory of G_C-projective R-modules,

$\mathrm{GF}_C(R)$ = the subcategory of G_C-flat R-modules.

Remark 1 From Example 2. 8(Holm et al. ,2006), if I is an injective R-module, then $\mathrm{Hom}_R(C, I)$ and I are G_C-injective, and if P is a projective R-module, then $C \otimes_R P$ and P are G_C-projective.

Similarly, if F is a flat R-module, then $C \otimes_R F$ and F are G_C-flat.

Definition 5 Let X be any class of R-modules and M an R-module. An X-precover of M is an R-homomorphism $\varphi : X \to M$, where $X \in$ X and such that the sequence

$$\mathrm{Hom}_R(X', X) \xrightarrow{\mathrm{Hom}_R(X', \varphi)} \mathrm{Hom}_R(X', M) \to 0$$

is exact for every $X' \in$ X. If, moreover, $\varphi \cdot f = \varphi$ for $f \in \mathrm{Hom}_R(X, X)$ implies f is an automorphism of X, then φ is called an X-cover of M.

2 Gorenstein homological modules and Auslander categories

Lemma 1(Di et al. ,2012) Let $\varphi : R \to S$ be a homomorphism of rings with $\mathrm{fd}_R(S) < +\infty$, C a semidualizing R-module. Then $\tilde{C} = C \otimes_R S$ is a semidualizing S-module.

Lemma 2 Let $\varphi : R \to S$ be a homomorphism of rings with $\mathrm{fd}_R(S) < +\infty$, C a semidualizing R-module and $\tilde{C} = C \otimes_R S$. If I_C is a C-injective R-module and $\tilde{F}$ is a flat S-module, then $\tilde{F} \otimes_R \mathrm{I}_C$ is a $\tilde{C}$-injective S-module.

Proof Consider the following isomorphisms:

$$\begin{aligned}\tilde{F} \otimes_R \mathrm{I}_C &\cong \tilde{F} \otimes_R \mathrm{Hom}_R(C, I) \cong \mathrm{Hom}_R(C, I \otimes_R \tilde{F}) \cong \mathrm{Hom}_R(C, \mathrm{Hom}_S(S, I \otimes_R \tilde{F}))\\ &\cong \mathrm{Hom}_S(C \otimes_R S, I \otimes_R \tilde{F}) \cong \mathrm{Hom}_S(\tilde{C}, I \otimes_R \tilde{F}),\end{aligned}$$

the second isomorphism holds by Theorem 3. 2. 14(Enochs et al. ,2000), the fourth by Hom-tensor adjointness, and $I \otimes_R \tilde{F}$ is an injective S-module by Proposition 2. 3(Christensen et al. ,2009).

Lemma 3 Let $\varphi : R \to S$ be a flat ring homomorphism, C a semidualizing R-module and $\tilde{C} = C \otimes_R S$. If M is a G_C-injective R-module, then $S \otimes_R M$ is a $G_{\tilde{C}}$-injective S-module.

Proof Since M is G_C-injective, by Proposition 2. 2 (White,2010), there exists a complete $\mathrm{I}_C\mathrm{I}$-resolution of M:

$$\Pi = \cdots \to \mathrm{Hom}_R(C, I_1) \to \mathrm{Hom}_R(C, I_0) \to I^0 \to I^1 \to \cdots$$

such that $\mathrm{Hom}_R(\mathrm{Hom}_R(C, E), \Pi)$ is exact for any injective R-module E.

Since S is flat as an R-module,

$$S \otimes_R \Pi = \cdots \to S \otimes_R \mathrm{Hom}_R(C, I_1) \to S \otimes_R \mathrm{Hom}_R(C, I_0) \to S \otimes_R I^0 \to S \otimes_R I^1 \to \cdots$$

is exact. It follows from Proposition 2. 3(Christensen et al. ,2009) and Lemma 2 that $S \otimes_R I^i$ are injective S-modules and $S \otimes_R \mathrm{Hom}_R(C, I_i)$ are $\tilde{C}$-injective S-modules and $S \otimes_R M \cong \mathrm{Ker}(S \otimes_R I^0 \to I \otimes_R I^1)$ by flatness of S. Furthermore, for any injective S-module $\tilde{J}$, it follows from Theorem3. 2. 14(Enochs et al. ,2000) and Hom-tensor adjointness, one has

$$\mathrm{Hom}_S(\mathrm{Hom}_S(S\otimes_R C,\tilde{J}),S\otimes_R \Pi)\cong \mathrm{Hom}_S(\mathrm{Hom}_S(S\otimes_R C,\tilde{J}),\Pi)\otimes_R S$$
$$\cong \mathrm{Hom}_S(\mathrm{Hom}_R(C,\tilde{J}),\Pi)\otimes_R S\cong \mathrm{Hom}_R(\mathrm{Hom}_R(C,\tilde{J}),\Pi)\otimes_R S$$

is exact.

Corollary 1 Let $\varphi: R\to S$ be a flat ring homomorphism, C a semidualizing R-module and $\tilde{C}=C\otimes_R S$. Then $\mathrm{GI}_{\tilde{C}}\text{-id}_S(S\otimes_R M)\leqslant \mathrm{GI}_C\text{-id}_R(M)$.

Lemma 4 (Sather-wagstaff et al., 2009) Let C be a semidualizing R-module, M an R-module with $\mathrm{GI}_C\text{-id}_R(M)<+\infty$. Then there is an exact sequence of R-modules

$$0\to M'\to E\to M\to 0$$

such that $\mathrm{I}_C\text{-id}_R(E)=\mathrm{GI}_C\text{-id}_R(M)$ and M' is a G_C-injective R-module.

Let (R,m,k) be a commutative noetherian local ring and M an R-module. We use the notation M^T for the Matlis dual $\mathrm{Hom}_R(M,E(k))$ of M, where $E(k)$ is the injective hull of the residue field k. There is a natural homomorphism $\varphi: M\to M^{TT}$ defined by $\varphi(x)(f)=f(x)$ for $x\in M$ and $f\in M^T$. It is well known that the canonical map φ is an embedding (Enochs et al. ,2000).

Recalling that an R-module M is Matlis reflexive if $M\cong M^{TT}$ under the canonical homomorphism $\varphi: M\to M^{TT}$. For instance, $\hat{R}$ is a Matlis reflexive R-module if R is complete, see Corollary 2. 5. 16 and Lemma 3. 4. 6 (Enochs et al. ,2000).

Proposition 1 Let R be complete. Then an R-module M is G_C-injective if and only if $\hat{R}\otimes_R M\in \mathrm{B}_{\hat{C}^+}(\hat{R})$ and $\mathrm{Ext}_R^{i\geqslant 1}(\mathrm{Hom}_R(C,E),M)=0$ for all injective R-modules E.

Proof Let M be a G_C-injective R-module. By Corollary 1, $\mathrm{GI}_{\hat{C}}\text{-id}_{\hat{R}}(\hat{R}\otimes_R M)<+\infty$.

Hence if follows from Theorem 4. 6(Holm et al., 2006) and Definition 4, that $\hat{R}\otimes_R M\in \mathrm{B}_{\hat{C}^+}(\hat{R})$ and $\mathrm{Ext}_R^{i\geqslant 1}(\mathrm{Hom}_R(C,E),M)=0$ for all injective R-modules E.

Conversely, it is enough to show that M admits a proper $\mathrm{I}_C(R)$-resolution; see Definition 4. Since $\hat{R}\otimes_R M\in \mathrm{B}_{\hat{C}^+}(\hat{R})$, one has $\mathrm{GI}_{\hat{C}}\text{-id}_{\hat{R}}(\hat{R}\otimes_R M)<+\infty$ by Theorem 4. 6(Holm et al. ,2006). It follows from Lemma 4 that there is an exact sequence of $\hat{R}$-modules

$$H\to \hat{R}\otimes_R M\to 0,$$

where $\mathrm{I}_{\hat{C}}\text{-id}_{\hat{R}}(H)=\mathrm{GI}_{\hat{C}}\text{-id}_{\hat{R}}(\hat{R}\otimes_R M)$.

Tensoring the above sequence with $\hat{R}$ yields the exact sequence $\hat{R}\otimes_R H\to \hat{R}\otimes_R\hat{R}\otimes_R M\to 0$. By Theorem 2(Belshoff, 1994), one has $\hat{R}\otimes_R\hat{R}\cong\hat{R}$ as R-module for $\hat{R}$ is a Matlis reflexive R-module.

Consequently, one has the exact sequence $H\to M\to 0$ for $\hat{R}$ is a faithfully flat R-module, where $\mathrm{I}_C\text{-id}_R(H)<+\infty$ for every $\hat{C}$-injective $\hat{R}$-module is C-injective as an R-module. It follows from Lemma 3. 3(2) (Zhang et al. ,2013) that there exists an epic $\overline{\mathrm{I}}_C(R)$-precover $E\to M$ with E C-injective.

Therefore, one has the following exact sequence

$$0\to B\to E\xrightarrow{f} M\to 0,$$

where f is an $\overline{\mathrm{I}}_C(R)$-precover and $B=\mathrm{Ker}\, f$.

Next we show that B satisfies the given assumptions on M.

Since f is an $\overline{\mathrm{I}}_C(R)$-precover, it is easy to see that $\mathrm{Ext}_R^{i\geqslant 1}(\mathrm{Hom}_R(C,I),B)=0$, where I is an injective R-module. Now it remains to prove that $\hat{R}\otimes_R B\in \mathrm{B}_{\hat{C}^+}(\hat{R})$.

As $\hat{R}$ is flat as R-module, one has the following exact sequence

$$0\to \hat{R}\otimes_R B\xrightarrow{\beta}\hat{R}\otimes_R E\to \hat{R}\otimes_R M\to 0.$$

By Lemma 2, $\hat{R}\otimes_R E$ is an $\hat{C}$-injective $\hat{R}$-module, so it is $\mathrm{G}_{\hat{C}}$-injective by Remark 1, then one has $\hat{R}\otimes_R B\in \mathrm{B}_{\hat{C}^+}(\hat{R})$ by the two-off-three property of Auslander categories. Now proceeding in this manner, one could get the desired proper $\mathrm{I}_C(R)$-resolution of M. This completes the proof.

Let $(R, \mathfrak{m}, k)$ be a local ring. Recall that the depth of an R-module M is defined as

$$\operatorname{depth}_R M = \inf\{i \mid \operatorname{Ext}_R^i(k, M) \neq 0\}.$$

The next result is an extension of the Bass equality for Gorenstein injective dimension over a local ring homomorphism (see Theorem 2. 3(Christensen et al. ,2019)) and for Gorenstein injective dimension with respect to a semidualizing module (see Proposition 4. 1(Salimi et al. , 2014b)), which is a special case by putting $C = R$ and $\varphi = \mathrm{id}_R$, respectively.

Lemma 5 Let $\varphi: R \to S$ be a local ring homomorphism and $N \neq 0$ a finitely generated S-module. If N has finite G_C-injective dimension over R, then one has

$$\mathrm{GI}_C\text{-id}_R(N) = \operatorname{depth} R = \mathrm{GI}_{\hat{C}}\text{-id}_{\hat{R}}(N \otimes_S \hat{S}).$$

Proof By Theorem 2. 16(Holm et al. ,2006), $\mathrm{GI}_C\text{-id}_R(N) = \mathrm{Gid}_{R \propto C} N$, where $R \propto C$ is the trivial extension of R by C. By Theorem 2. 5(Khatami et al. ,2009), $\mathrm{Gid}_{R \propto C} N = \operatorname{depth}(R \propto C)$ since $\mathrm{Gid}_{R \propto C} N < +\infty$. Note that $\operatorname{depth}(R \propto C) = \min\{\operatorname{depth} R, \operatorname{depth}_R C\} = \operatorname{depth} R$, since $\operatorname{depth}_R C = \operatorname{depth} R$ by Theorem 2. 2. 6(Sather-wagstaff, 2013).

Additionally, by Theorem 2. 3(Christensen et al. , 2019), one has

$$\mathrm{Gid}_{R \propto C} N = \mathrm{Gid}_{\widehat{R \propto C}}(N \otimes_{S \propto \hat{C}} \widehat{(S \propto \tilde{C})}) = \mathrm{Gid}_{\hat{R} \propto \hat{C}}(N \otimes_{S \propto \hat{C}} (\hat{S} \propto \hat{\hat{C}})) = \mathrm{Gid}_{\hat{C}}\text{-id}_{\hat{R}}(N \otimes_S \hat{S}).$$

Theorem 1 If R has a dualizing complex and M is a finitely generated R-module, then M is a G_C-injective R-module if and only if $\hat{R} \otimes_R M$ is a $G_{\hat{C}}$-injective $\hat{R}$-module.

Proof Since R has a dualizing complex, by Proposition 5. 9(Christensen, 2001), one has $M \in \mathrm{B}_{C^+}(R)$ if and only if $\hat{R} \otimes_R M \in \mathrm{B}_{\hat{C}^+}(\hat{R})$. Now the result follows from Theorem 4. 6(Holm et al. , 2006) and Lemma 5.

The following result is one of the main results in this paper which generalizes Theorem 2. 1(Wu et al. , 2016).

Theorem 2 An R-module M is G_C-projective if and only if the Matlis dual $M^T \in \mathrm{B}_{\hat{C}^+}(\hat{R})$ and $\operatorname{Ext}_R^{i \geqslant 1}(M, C \otimes_R P) = 0$ for all projective R-modules P.

Proof Let M be a G_C -projective R-module. By Corollary 4. 14(Di et al. , 2012), M^T is a $G_{\hat{C}}$-injective $\hat{R}$-module. Hence it follows from Theorem 4. 6(Holm et al. , 2006) and Definition 4, that the Matlis dual $M^T \in \mathrm{B}_{\hat{C}^+}(\hat{R})$ and $\operatorname{Ext}_R^{i \geqslant 1}(M, C \otimes_R P) = 0$ for all projective R-modules P.

Conversely, it is enough to show that M admits a proper $\mathrm{P}_C(R)$-coresolution; see Definition 4. Since $M^T \in \mathrm{B}_{\hat{C}^+}(\hat{R})$, $\mathrm{GI}_{\hat{C}}\text{-id}_{\hat{R}}(M^T) < +\infty$ by Theorem 4. 6(Holm et al. , 2006). It follows from Lemma 4 that there is an exact sequence of $\hat{R}$-modules $H \to M^T \to 0$ such that

$$\mathrm{I}_{\hat{C}}\text{-id}_{\hat{R}}(H) = \mathrm{GI}_{\hat{C}}\text{- id}_{\hat{R}}(M^T).$$

An application of $\mathrm{Hom}_R(-, E(k))$ yields the exact sequence $0 \to M^{TT} \to H^T$. By Proposition 5. 2(Salimi et al. , 2014a) and Theorem 4. 5(I)(Avramov et al. , 1991),one has $\mathrm{F}_{\hat{C}}\text{-pd}_{\hat{R}}(H^T) \leqslant \mathrm{I}_{\hat{C}}\text{-id}_{\hat{R}}(H)$ and so $\mathrm{F}_{\hat{C}}\text{-pd}_{\hat{R}}(H^T) < +\infty$. Since every $\hat{C}$-flat $\hat{R}$-module is also C-flat as an R-module, one has $\mathrm{F}_C\text{-pd}_{\hat{R}}(H^T) < +\infty$.

Consequently, there exists a monomorphism $\alpha: M \to H^T$ with $\mathrm{F}_C\text{-pd}_{\hat{R}}(H^T) < +\infty$. By Proposition 5. 3(d) (Holm et al. , 2007), there is a C-flat preenvelope $f: M \to F$.

Next we show that f is an $\overline{\mathrm{F}}_C(R)$-preenvelope. Let $\psi: M \to L$ be an R-homomorphism such that F_C-$\mathrm{pd}_R(L) < +\infty$ and let $0 \to K \to F' \to L \to 0$ be an exact sequence such that $\pi: F' \to L$ is a C-flat cover, see Proposition 5. 3(a)(Holm et al. , 2007). Clearly, K is of finite C-flat dimension and so K is of finite C-projective dimension. By hypothesis and induction on C-projective dimension of K, one has $\operatorname{Ext}_R^{i \geqslant 1}(M, K) = 0$. Thus, one has the following exact sequence

$$0 \to \mathrm{Hom}_R(M, K) \to \mathrm{Hom}_R(M, F') \to \mathrm{Hom}_R(M, L) \to 0.$$

Therefore, there exists an R-homomorphism $h: M \to F'$ such that $\pi h = \psi$. Since $f: M \to F$ is a C-flat preenvelope, there is an R-homomorphism $g: F \to F'$ such that $h = gf$. Thus, one has $\pi gh = \psi$ and so f is an $\overline{\mathrm{F}}_C(R)$-preenvelope. Hence there exists an R-homomorphism $\theta: F \to H^T$ such that $\theta f = \alpha$. Note that f is monic for α is a monomorphism.

Next we show that there exists a monic $\overline{\mathrm{P}}_C(R)$-preenvelope $M \to P$ with P C-projective. It is easy to see f is also a $\overline{\mathrm{P}}_C(R)$-preenvelope. Notice that $f: M \to F$ is monic as $\alpha: M \to H^T$ is monic and $\mathrm{P}_C\text{-pd}_R(H^T) < +\infty$. Let $0 \to A \to P \to F \to 0$ be an exact sequence with P C-projective. Clearly, one has $\mathrm{P}_C\text{-pd}_R(A) < +\infty$. It is easy to see that $\mathrm{Ext}_R^{i\geqslant 1}(M, A) = 0$. Therefore, there exists a monic $\overline{\mathrm{P}}_C(R)$-preenvelope $M \to P$ with P C-projective.

Now consider the following exact sequence

$$0 \to M \xrightarrow{\beta} P \to U \to 0.$$

where β is a $\overline{\mathrm{P}}_C(R)$-preenvelope, P is a C-projective R-module and $U = \mathrm{Coker}\,\beta$.

Let Q be a projective R-module. Applying the functor $\mathrm{Hom}_R(-, C \otimes_R Q)$ to the above exact sequence, one has $\mathrm{Ext}_R^{i\geqslant 1}(U, C \otimes_R Q) = 0$ for $\beta: M \to P$ is a $\overline{\mathrm{P}}_C(R)$-preenvelope. It is not hard to see that $U^T \in \mathrm{B}_{\hat{C}^+}(\hat{R})$. Now proceeding in this manner, one could get the desired proper $\mathrm{P}_C(R)$ -coresolution of M. This completes the proof.

The next result generalizes Theorem 2. 2(Sather-wagstaff, 2013) by putting $C = R$.

Theorem 3　An R-module M is G_C -flat if and only if the Matlis dual $M^T \in \mathrm{B}_{\hat{C}^+}(\hat{R})$ and $\mathrm{Tor}^R_{i\geqslant 1}(\mathrm{Hom}_R(C, I), M) = 0$ for all injective R-modules I.

Proof　Let M be a G_C -flat R-module. By Corollary 4. 12(2)(Takahashi et al. , 2010), M^T is a $G_{\hat{C}}$-injective $\hat{R}$-module. Hence it follows from Theorem 4. 6(White, 2010) and Definition 4, that the Matlis dual $M^T \in \mathrm{B}_{\hat{C}^+}(\hat{R})$ and $\mathrm{Tor}^R_{i\geqslant 1}(\mathrm{Hom}_R(C, I), M) = 0$ for all injective R-modules I.

Conversely, it is enough to show that M admits an $\mathrm{I}_C(R) \otimes_R$-exact $\mathrm{F}_C(R)$-coresolution; see Definition 4. By analogy with the proof of Theorem 2, there exists a monic $\overline{\mathrm{F}}_C(R)$-preenvelope $f: M \to F$ with F C-flat. Hence one has the following exact sequence

$$0 \to M \xrightarrow{f} F \to U \to 0,$$

where $U = \mathrm{Coker} f$. If F' is a C-flat R-module, then one has the exact sequence

$$0 \to \mathrm{Hom}_R(U, F') \to \mathrm{Hom}_R(F, F') \to \mathrm{Hom}_R(M, F') \to 0$$

for f is an $\overline{\mathrm{F}}_C(R)$-preenvelope. Since $\mathrm{Hom}_R(C, I)^T$ is a C-flat R-module, the next sequence

$$0 \to \mathrm{Hom}_R(U, \mathrm{Hom}_R(C, I)^T) \to \mathrm{Hom}_R(F, \mathrm{Hom}_R(C, I)^T) \to \mathrm{Hom}_R(M, \mathrm{Hom}_R(C, I)^T) \to 0$$

is exact. By adjointness, one also has the following exact sequence

$$0 \to (U \otimes_R \mathrm{Hom}_R(C, I))^T \to (F \otimes_R \mathrm{Hom}_R(C, I))^T \to (M \otimes_R \mathrm{Hom}_R(C, I))^T \to 0.$$

Since $E(k)$ is an injective cogenerator, one has the next exact sequence

$$0 \to M \otimes_R \mathrm{Hom}_R(C, I) \to F \otimes_R \mathrm{Hom}_R(C, I) \to U \otimes_R \mathrm{Hom}_R(C, I) \to 0.$$

Consequently, one has $\mathrm{Tor}^R_{i\geqslant 1}(\mathrm{Hom}_R(C, I), U) = 0$ as $\mathrm{Tor}^R_{i\geqslant 1}(\mathrm{Hom}_R(C, I), M) = 0$. It is not hard to see that $U^T \in \mathrm{B}_{\hat{C}^+}(\hat{R})$. Now proceeding in this manner, one could get the desired $\mathrm{I}_C(R) \otimes_R$-exact $\mathrm{F}_C(R)$ -coresolution of M. This completes the proof.

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相对于半对偶模的Gorenstein同调模与Auslander范畴

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摘要： 用交换noetherian局部环R上的Auslander范畴给出了相对于半对偶R-模C的Gorenstein内射、投射和平坦模的一些新刻画。

关键词： 半对偶模；Auslander范畴；G_C-投射(内射,平坦)模

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