DOI：10. 13471/j. cnki. acta. snus. ZR20250163

Dynamics of the perturbed Calogero-Moser system: well-posedness, stability and blow up*

LIU Qianle, WANG Zhong, ZHU Weipeng ✉

School of Mathematics, Foshan University, Foshan 528000, China

Abstract: We investigate a class of non-integrable two-particle Calogero-Moser systems modulated by a power-law external potential. The local well-posedness of the Cauchy problem is established under the strict initial separation condition for the particles. For suitably prepared initial configurations, local solutions can be extended globally via energy conservation; conversely, negative energy conditions induce (in)finite-time blowup. The linear (in)stability of stationary solutions is analyzed, with their energy serving as a threshold. Numerical investigations employ a fourth-order Runge-Kutta scheme with adaptive step-size control. Simulations demonstrate that the trajectories either converge to steady states or exhibit blowup, depending on the power exponent α and initial conditions. Increasing α accelerates the convergence rate and dampens oscillatory dynamics, promoting a transition from periodic behavior to static equilibrium.

Key words: Calogero-Moser system; well-posedness; blow up; stability

CLC number: O175.13　　**Document code:** A　　**Article ID:** 2097 – 0137(2026)01 – 0157 – 12

The nonlinear wave dynamics of long-range interacting particle systems have been a significant research area in mathematical physics and condensed matter physics. The Calogero-Moser system, as a typical integrable many-body model, has attracted extensive attention due to its elegant algebraic structure and rich physical implications (Calogero,1971;Sutherland,1971;Morse,1975). Originally proposed independently by Calogero and Moser, its dynamical equation is:

$$\ddot{x}_j = 2\sum_{k \neq j}\left(x_j - x_k\right)^{-3},\ x_j = x_j(t) \in \mathbb{R},\ t > 0,\ j,k = 1,2,\cdots,N,\ N > 0, \tag{CM}$$

which is comprised of N interacting bodies in one spatial dimension via an inverse cubic force $F \propto \left(x_j - x_k\right)^{-3}$, with the added virtue of being completely integrable (Olshanetsky et al. ,1981;Polychronakos,1995). The solitons of the system and deep connections with nonlinear partial differential equations (such as the Benjamin-Ono equation) have provided a paradigm for understanding the dynamics of solutions in one-dimensional nonlinear lattices (Case,1978;Stone et al. ,2008; Abanov et al. ,2009).

Recently, the dynamics of infinite-range power-law interacting particle systems have emerged as a prominent research focus in the field. Ingimarson et al. (2024a, 2024b) theoretically established that when the interaction potential adheres to the form $F \propto \left(x_j - x_k\right)^{-\alpha}$, $\alpha > 0$, the system can be reduced to either a generalized

---

* **Received**: 2025 – 07 – 28　　**Accepted**: 2025 – 08 – 11　　**Published online**: 2025 – 10 – 23

Supported by National Natural Science Foundation of China（12201118）;
Guangdong Basic and Applied Basic Research Foundation（2023A1515010706）

✉ **Corresponding author**: ZHU Weipeng（*zhuweipeng@fosu.edu.cn*）

LIU Qianle（*1321244430@qq.com*）; WANG Zhong（*wangzh79@fosu.edu.cn*）

Benjamin-Ono equation or the classical KdV equation in the long-wave limit, contingent upon the power exponent α. Their investigations also delved into soliton structures governed by power-law forces. Notably, the case of $\alpha = 3$ corresponds to the infinite-particle Calogero-Moser lattice, whose solitary wave solutions can be explicitly constructed via Bäcklund transformations, as documented in Wojciechowski(1982). However, real-world physical systems are frequently modulated by external potentials. For instance, in cold atom chains (Molerón et al., 2014) and magnetic spin lattices (English et al., 2005), particles are typically subjected to harmonic confinements, which substantially alter the system dynamics. While the introduction of external potentials may disrupt classical integrability, it can also give rise to novel nonlinear phenomena, including localized mode bifurcations and energy level quantization, as discussed in Flach et al. (2008).

In this paper, we consider the Cauchy problems of the perturbed Calogero-Moser system as follows:

$$\begin{cases} \ddot{x}_1 = 2\left(x_1 - x_2\right)^{-\alpha-1} + \gamma x_1^{\beta}, \\ \ddot{x}_2 = 2\left(x_2 - x_1\right)^{-\alpha-1} + \gamma x_2^{\beta}, \\ x_j = x_j(t) \in \mathbb{R},\ j = 1, 2, \\ x_1(0) = a, \quad x_2(0) = b, \quad \dot{x}_1(0) = c, \quad \dot{x}_2(0) = d,\ \ a, b, c, d \in \mathbb{R}, \end{cases} \tag{PCM}$$

where integers $\alpha, \beta > 0$ are the power-law exponents, x_1^{β} and x_2^{β} are the external potentials and $\gamma = \pm 1$. When $\alpha = 2$, the interaction term degenerates into the inverse-cubic potential of the classical Calogero-Moser system, making this system a two-particle Calogero-Moser model modulated by an external potential.

For the rational Calogero-Moser system (CM) subject to harmonic confinement, its integrability enables the derivation of exact solutions via Lax pair formulations and Bäcklund transformations (Abanov et al. ,2011;Philip, 2019). Specifically, the system's algebraic structure under such potentials allows for systematic solution constructions through classical inverse scattering techniques. However, the introduction of nonlinear perturbations such as pure power-law terms $x_j^{\beta}, j = 1,2$ renders the modified system (PCM) non-integrable. This non-integrability fundamentally disrupts the applicability of classical techniques for integrable systems, as the Lax representation and Bäcklund transform methodologies no longer hold. Confronted with this non-integrable regime, alternative investigative approaches are necessary. In the absence of explicit closed-form solutions, the research focus shifts to establishing solution properties through analytical frameworks, prioritizing the study of existence, uniqueness, and dynamical behaviors. By integrating analytical perturbation methods with numerical simulations to attain a comprehensive characterization of non-integrable Hamiltonian systems.

This paper explores the nonlinear dynamics of finite energy solutions of (PCM), namely, the energy of (PCM) satisfies

$$E(t) := E\left(x_1, x_2\right) = \frac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) + \frac{2}{\alpha(x_1 - x_2)^{\alpha}} - \frac{\gamma}{\beta + 1}\left(x_1^{\beta+1} + x_2^{\beta+1}\right) < +\infty. \tag{1}$$

Firstly, we show the following local well-posedness (existence, uniqueness and continuous dependence of solutions) under the initial separation condition.

Theorem 1 (Local well-posedness) Consider the Cauchy problem of (PCM) with

(i) The initial separation condition: $\left|x_1(0) - x_2(0)\right| \geqslant \delta > 0$;

(ii) Solution restricted to the closed region:

$$S = \left\{\left(t, x_1, x_2, \dot{x}_1, \dot{x}_2\right) \,\middle|\, |t| \leqslant T,\ \left|x_1 - a\right| \leqslant A,\ \left|x_2 - b\right| \leqslant B,\ \left|\dot{x}_1 - c\right| \leqslant C,\ \left|\dot{x}_2 - d\right| \leqslant D,\ \left|x_1 - x_2\right| \geqslant \frac{\delta}{2}\right\}.$$

Then, the initial value problem (PCM) has a unique solution on the interval $\left[-h, h\right]$, where

$$h = \min\left\{T, \frac{1}{M}\min\{A, B, C, D\}\right\},$$

and

$$M = \max\left\{|c| + C, |d| + D, 2(\delta/2)^{-\alpha-1} + \left(|a| + A\right)^{\beta}, 2(\delta/2)^{-\alpha-1} + \left(|b| + B\right)^{\beta}\right\}.$$

Moreover, the solution continuously depends on the initial data $\left(a, b, c, d\right)$.

Subsequently, leveraging the energy conservation law, the total energy (1) of system (PCM) is proven to be conserved, thereby establishing the boundedness of particle positions and velocities. This conservation property facilitates the extension of local solutions to global ones, guaranteeing their existence over the entire time domain.

Theorem 2 (Global existence) The Cauchy problem of (PCM) with even α, odd β, $\gamma = -1$ and the initial separation condition $\left|x_1(0) - x_2(0)\right| \geqslant \delta > 0$ is globally well-posed in $\mathbb{R}$.

Corollary 1 (Liouville property) The Cauchy problem of (PCM) with even α, odd β and $\gamma = -1$ has the following properties:

(i) The energy of the global solution satisfies $0 < E_{\alpha,\beta} \leqslant E\left(x_1, x_2\right) < +\infty$, where

$$E_{\alpha,\beta} := E\left(2^{-\frac{\alpha}{\alpha+\beta+1}}, 2^{-\frac{\alpha}{\alpha+\beta+1}}\right) = \left(\frac{1}{\alpha} + \frac{1}{\beta+1}\right)2^{\frac{\beta+1-\alpha\beta}{\alpha+\beta+1}}. \tag{2}$$

(ii) If the initial energy satisfies $E\left(x_1, x_2\right) = E_{\alpha,\beta}$, then the global solutions must be constant, namely,

$$x_1(t) = -x_2(t) \equiv \pm 2^{-\frac{\alpha}{\alpha+\beta+1}}. \tag{3}$$

However, the global well-posedness of the Cauchy problem of (PCM) for other cases is more involved. For example, when α is odd, β is even and $\gamma = -1$, if the initial conditions satisfy $x_1(0) = -x_2(0)$ and $\dot{x}_1(0) = \dot{x}_2(0) = 0$, then the solutions are

$$x_1(t) = 2^{-\frac{\alpha}{\alpha+\beta+1}}, \quad x_2(t) = -2^{-\frac{\alpha}{\alpha+\beta+1}}, \tag{4}$$

or

$$x_1(t) = -2^{-\frac{\alpha}{\alpha+\beta+1}}, \quad x_2(t) = 2^{-\frac{\alpha}{\alpha+\beta+1}}, \tag{5}$$

which are global and time independent. However, since β is odd, the associated energies are $\frac{1}{\alpha}2^{\frac{\beta+1-\alpha\beta}{\alpha+\beta+1}}$ and $-\frac{1}{\alpha}2^{\frac{\beta+1-\alpha\beta}{\alpha+\beta+1}}$, respectively.

Remark 1 Corollary 1 states that $E_{\alpha\beta}$ is the energy threshold for the Cauchy problem of (PCM) with even α, odd β and $\gamma = -1$. However, for the case of odd α, even β and $\gamma = -1$, there exists no energy threshold.

Solutions (3), (4) and (5) are stationary solutions of (PCM), we are interested in their stability.

Theorem 3 (Stability and instability) Solutions (3) are linearly stable, solutions (4) and (5) are linearly unstable.

In the rest of this section, we consider only the case $\alpha = 2$, even β and $\gamma = -1$ of (PCM) and investigate the blow up of the solutions.

Theorem 4 (Infinite time blow up) Consider the Cauchy problem of (PCM). Suppose that the following conditions hold:

(i) Initial total energy $E(0) < -\varepsilon < 0$;

(ii) Strict separation $\left|x_1(0) - x_2(0)\right| \geqslant \delta > 0$;

Then the solution of (PCM) blows up at $t \to +\infty$, i. e. ,

$$\lim_{t \to +\infty}\left(\left|x_1(t)\right| + \left|x_2(t)\right|\right) = +\infty.$$

Corollary 2 (Finite time blow up) Let $x_j = x_j(t) \in \mathbb{R}$, $t > 0, j = 1,2$. Consider the system

$$\begin{cases} \ddot{x}_1 = -2\left(x_1 - x_2\right)^{-3}, \\ \ddot{x}_2 = -2\left(x_2 - x_1\right)^{-3}, \end{cases}$$

then the energy reduces to

$$E(t) := E(x_1, x_2) = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) - (x_1 - x_2)^{-2}.$$

In this case, if we choose $E(0) < 0$, then the solutions exhibit finite-time blow-up, i. e. , there exists some time $T_{max} > 0$ such that

$$\lim_{t \to T_{max}} \left(|\dot{x}_1(t)| + |\dot{x}_2(t)|\right) = +\infty.$$

Remark 2 The results of Theorem1, Theorem 2, Theorem 4 and Corollary 2 hold for systems of N particles as well. The proof method remains the same; we only present the two-particle case here for clarity.

Indeed, the perturbed Calogero-Moser system with N particles as follows:

$$\ddot{x}_j = \sum_{k \neq j} 2(x_j - x_k)^{-\alpha - 1} + \gamma x_j^\beta, \quad x_j = x_j(t) \in \mathbb{R}, t > 0, \ j, k = 1,2,\cdots,N, \ N > 0. \tag{N-PCM}$$

The total energy of (N-PCM) can be expressed as:

$$E(t) := E(x_1, x_2, \cdots, x_N) = \sum_{j=1}^{N} \frac{1}{2} \dot{x}_j^2 + V(x_1, x_2, \cdots, x_N),$$

which is conserved for all $t \in \mathbb{R}$, where the potential energy function is:

$$V(x_1, x_2, \cdots, x_N) := \sum_{j=1}^{N-1} \sum_{j<k} \frac{2}{\alpha (x_j - x_k)^\alpha} - \frac{\gamma}{\beta + 1} \sum_{j=1}^{N} x_j^{\beta + 1}.$$

Consider the systems of N particles

$$\ddot{x}_j = -\sum_{k \neq j} 2(x_j - x_k)^{-3}, \quad x_j = x_j(t) \in \mathbb{R}, t > 0, \ j, k = 1,2,\cdots,N, \ N > 0. \tag{N-CM}$$

Then the energy reduces to

$$E(t) := E(x_1, x_2, \cdots, x_N) = \sum_{j=1}^{N} \frac{1}{2} \dot{x}_j^2 - \sum_{j=1}^{N-1} \sum_{j<k} (x_j - x_k)^{-2}.$$

Finally, long-time numerical integrations of system (PCM) with $\beta = 2$ are performed via the fourth-order Runge-Kutta method. The numerical results demonstrate that under varying power-law exponents α and initial conditions, the system trajectories exhibit either asymptotic stabilization or finite-time blowup, with stable configurations showing insensitivity to further time evolution. An increase in α accelerates the decay of inter-particle interactions, thereby transitioning the system from periodic oscillatory modes to static equilibrium states. Conversely, smaller initial inter-particle separations or larger initial velocities induce transient chaotic regimes characterized by aperiodic energy redistribution.

1 Local well-posedness

In this section, we give the proof of Theorem 1.

Proof Define $\boldsymbol{X} = (x_1, x_2, \dot{x}_1, \dot{x}_2)^{\mathrm{T}}$, then the system (PCM) can be rewritten as:

$$\begin{cases} \dfrac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \boldsymbol{F}(\boldsymbol{X}), \\ \boldsymbol{X}(0) = (a, b, c, d)^{\mathrm{T}}, \end{cases} \tag{6}$$

where $\boldsymbol{F}(\boldsymbol{X}) := \left(\dot{x}_1, \dot{x}_2, 2(x_1 - x_2)^{-\alpha - 1} + \gamma x_1^\beta, 2(x_2 - x_1)^{-\alpha - 1} + \gamma x_2^\beta\right)^{\mathrm{T}}$. It is easy to see that $\boldsymbol{F}(\boldsymbol{X})$ is Lipschitz continuous with respect to $\boldsymbol{X}$ in the closed region S. Indeed, $\boldsymbol{F}(\boldsymbol{X})$ is continuous in S and satisfying $|\boldsymbol{F}(\boldsymbol{X}^1) - \boldsymbol{F}(\boldsymbol{X}^2)| \leqslant L|\boldsymbol{X}^1 - \boldsymbol{X}^2|$, for any $(t, \boldsymbol{X}^i) \in S, i = 1,2$, where

$$L = 4(\alpha + 1)(\delta/2)^{-\alpha - 2} + \beta(|a| + A)^{\beta - 1} + \beta(|b| + B)^{\beta - 1} + 2.$$

Note that $|\boldsymbol{F}(\boldsymbol{X})| \leqslant M$, where

$$M=\max\left\{|c|+C,\ |d|+D,\ 2(\delta/2)^{-\alpha-1}+\left(|a|+A\right)^{\beta},\ 2(\delta/2)^{-\alpha-1}+\left(|b|+B\right)^{\beta}\right\}.$$

Define $h=\min\left\{T,\frac{1}{M}\min\{A,B,C,D\}\right\}$, then by the Picard existence and uniqueness theorem, the initial value problem (PCM) has a unique solution on the interval $[-h,h]$.

Next, we prove the solution is continuous dependence on the initial data. It is well known the system (6) is equivalent to the following integral equation:

$$\boldsymbol{X}(t)=\boldsymbol{X}(0)+\int_0^t\boldsymbol{F}\left(\boldsymbol{X}(s)\right)\mathrm{d}s.$$

Let $\boldsymbol{X}^1(t)$ and $\boldsymbol{X}^2(t)$ be the solutions corresponding to the initial data $\boldsymbol{X}^1(0)$ and $\boldsymbol{X}^2(0)$. Then

$$\left|\boldsymbol{X}^1(t)-\boldsymbol{X}^2(t)\right|\leqslant\left|\boldsymbol{X}^1(0)-\boldsymbol{X}^2(0)\right|+\left|\int_0^t\boldsymbol{F}\left(\boldsymbol{X}^1(s)\right)-\boldsymbol{F}\left(\boldsymbol{X}^2(s)\right)\mathrm{d}s\right|\leqslant\left|\boldsymbol{X}^1(0)-\boldsymbol{X}^2(0)\right|+L\int_0^t\left|\boldsymbol{X}^1(s)-\boldsymbol{X}^2(s)\right|\mathrm{d}s.$$

Therefore, by Gronwall's inequality, we have

$$\left|\boldsymbol{X}^1(t)-\boldsymbol{X}^2(t)\right|\leqslant\left|\boldsymbol{X}^1(0)-\boldsymbol{X}^2(0)\right|\mathrm{e}^{L|t|},$$

which implies that the solution is continuous dependence on the initial data. This completes the proof of Theorem 1.

2 Global existence and Liouville property

This section is devoted to prove Theorem 2 and Corollary 1, namely, the Cauchy problem of (PCM) with even α, odd β, $\gamma=-1$ is globally well posed.

Lemma 1 The total energy of (PCM) can be expressed as:

$$E(t):=E(x_1,x_2)=\frac{1}{2}\left(\dot{x}_1^2+\dot{x}_2^2\right)+V(x_1,x_2),$$

which is conserved for all $t\in\mathbb{R}$, where the potential energy function is:

$$V(x_1,x_2):=\frac{2}{\alpha(x_1-x_2)^{\alpha}}+\frac{1}{\beta+1}\left(x_1^{\beta+1}+x_2^{\beta+1}\right).$$

Proof The time derivatives with respect to kinetic and potential energy are

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{x}_1^2+\dot{x}_2^2\right)=\dot{x}_1\ddot{x}_1+\dot{x}_2\ddot{x}_2$$

and

$$\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\partial V}{\partial x_1}\dot{x}_1+\frac{\partial V}{\partial x_2}\dot{x}_2,$$

respectively. Thus, the time derivative of the total energy is:

$$\frac{\mathrm{d}E}{\mathrm{d}t}=\dot{x}_1\ddot{x}_1+\dot{x}_2\ddot{x}_2+\frac{\partial V}{\partial x_1}\dot{x}_1+\frac{\partial V}{\partial x_2}\dot{x}_2. \tag{7}$$

Substituting the equation of motion $\ddot{x}_i=-\partial_{x_i}V$ into (7):

$$\frac{\mathrm{d}E}{\mathrm{d}t}=\sum_{i=1}^{2}\dot{x}_i\left(-\partial_{x_i}V\right)+\sum_{i=1}^{2}\left(\partial_{x_i}V\right)\dot{x}_i=0.$$

The proof is completed.

We are in a position to give the proof of Theorem 2.

Proof We prove this result by contradiction. Suppose that the maximum time of existence for the solution $T_{\max}<+\infty$. It reveals from energy conservation $E(t)=E(0)$ and kinetic energy $T(t):=\frac{1}{2}\left(\dot{x}_1^2+\dot{x}_2^2\right)\geqslant 0$ that

$$E(t)\geqslant V(x_1,x_2)=\frac{2}{\alpha(x_1-x_2)^{\alpha}}+\frac{1}{\beta+1}\left(x_1^{\beta+1}+x_2^{\beta+1}\right)\geqslant\frac{2}{\alpha\left|x_1-x_2\right|^{\alpha}}.$$

We then have

$$\left|x_1(t)-x_2(t)\right| \geqslant \left(\frac{2}{\alpha E(0)}\right)^{1/\alpha} := \delta_0 > 0, \quad \forall t \in [0, T_{\max}).$$

Since

$$\frac{1}{2}\left(\dot{x}_1^2+\dot{x}_2^2\right) \leqslant E(0) \Rightarrow \left|\dot{x}_j(t)\right| \leqslant \sqrt{2E(0)},\ j=1,2.$$

Furthermore, one infers from the potential energy term $\frac{x_j^{\beta+1}}{\beta+1} \leqslant E(0)$ that

$$\left|x_j(t)\right| \leqslant \left((\beta+1)E(0)\right)^{\frac{1}{\beta+1}} < +\infty,\ j=1,2, \quad \forall t \in [0, T_{\max}).$$

Then, as $t \to T_{\max}^-$, the solution $\left(x_1(t), x_2(t)\right)$ satisfies:

(i) Position is bounded: $\left|x_j(t)\right| < +\infty$;

(ii) Velocity is bounded: $\left|\dot{x}_j(t)\right| \leqslant \sqrt{2E(0)}$;

(iii) $\left|x_1(t)-x_2(t)\right| \geqslant \delta_0$.

The local well-posedness statement in Theorem 1 reveals that there exists $\epsilon > 0$, such that the solution can be extended to $[0, T_{\max}+\epsilon)$, contradicting the maximality of $T_{\max}$. Hence, $T_{\max}=+\infty$ and the solution is global.

Now we are in position of showing Corollary 1. Recall that the energy of the system (with even α and odd $\beta=2k+1$)

$$E\left(x_1,x_2\right)=\frac{1}{2}\left(\dot{x}_1^2+\dot{x}_2^2\right)+\frac{2}{\alpha\left(x_1-x_2\right)^{\alpha}}+\frac{1}{2k+2}\left(x_1^{2k+2}+x_2^{2k+2}\right).$$

We investigate whether there exists a critical energy value $E_{\min}$ such that solutions cannot exist globally for $E < E_{\min}$, but can exist globally for $E \geqslant E_{\min}$. Specifically, we verify whether the energy corresponding to the equilibrium solutions (3), which represents this minimum global energy threshold.

Since kinetic energy is non-negative, the lower bound of the total energy is controlled by the minimum of the potential energy. To the aim of

$$\min_{x_1,x_2\in\mathbb{R}, x_1\neq x_2} V\left(x_1,x_2\right),$$

we derive the following two extremum conditions

$$\begin{cases} \dfrac{\partial V}{\partial x_1}=-2\left(x_1-x_2\right)^{-\alpha-1}+x_1^{2k+1}=0, \\ \dfrac{\partial V}{\partial x_2}=2\left(x_1-x_2\right)^{-\alpha-1}+x_2^{2k+1}=0. \end{cases} \tag{8}$$

Adding the two above equations gives

$$x_1^{2k+1}+x_2^{2k+1}=0 \Rightarrow x_1=-x_2\neq 0.$$

Therefore, (8) infers that

$$2\left(2x_1\right)^{-\alpha-1}=x_1^{2k+1}, \quad x_1=\pm 2^{-\frac{\alpha}{\alpha+\beta+1}}, \quad \beta=2k+1,$$

which gives the minimum of the potential energy

$$V_{\min}=\left(\frac{1}{\alpha}+\frac{1}{\beta+1}\right)2^{\frac{\beta+1-\alpha\beta}{\alpha+\beta+1}}=E_{\alpha,\beta}.$$

At this point, kinetic energy is zero (equilibrium solution), thus,

$$E_{\min}=V_{\min}=\left(\frac{1}{\alpha}+\frac{1}{\beta+1}\right)2^{\frac{\beta+1-\alpha\beta}{\alpha+\beta+1}}=E_{\alpha,\beta}. \tag{9}$$

Proof of Corollary 1 The Cauchy problem of (PCM) with even α and odd $\beta=2k+1$ and (9) indicate that the energy of which satisfies

$$E\left(x_1,x_2\right) \geqslant E_{\min}=\left(\frac{1}{\alpha}+\frac{1}{\beta+1}\right)2^{\frac{\beta+1-\alpha\beta}{\alpha+\beta+1}}=E_{\alpha,\beta}>0,$$

all finite energy solutions of (PCM) are global, no solutions of (PCM) verify $E\left(x_1, x_2\right) < E_{\min}$.

Now we show the second statement. If the initial energy satisfies $E\left(x_1, x_2\right) = E_{\alpha,\beta}$, then (8) reveals that the kinetic energy must be zero, namely, $\dot{x}_1 = \dot{x}_2 = 0$, which indicates that the solutions are constants (3). The proof of the Corollary 1 is completed.

3　Stability and instability

In this Section, we study the dynamics (stability issue) of four stationary solutions (3), (4) and (5) of (PCM) and finish the proof of Theorem 3. Essentially, there are two constant solutions of (PCM) under different parameters α and β. We denote the two constant solutions by

$$A_1 = -A_2 = \pm 2^{-\frac{\alpha}{\alpha+\beta+1}}. \tag{10}$$

We linearize the (PCM) around the constant solutions (10) and study the eigenvalues of the corresponding Jacobian matrix.

Proof of Theorem 3　Let us introduce the variables

$$\boldsymbol{X} = \left(x_1, x_2, \dot{x}_1, \dot{x}_2\right)^{\mathrm{T}}, \qquad \frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \boldsymbol{F}(\boldsymbol{X}),$$

where

$$\boldsymbol{F}(\boldsymbol{X}) = \left(\dot{x}_1, \dot{x}_2, 2\left(x_1 - x_2\right)^{-\alpha-1} - x_1^{\beta}, 2\left(x_2 - x_1\right)^{-\alpha-1} - x_2^{\beta}\right)^{\mathrm{T}} = \left(f_1, f_2, f_3, f_4\right)^{\mathrm{T}}.$$

Then the Jacobian matrix $\boldsymbol{J} = D\boldsymbol{F}(\boldsymbol{X})$ has the following structure

$$\boldsymbol{J} = \left.\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{\partial f_3}{\partial x_1} & \dfrac{\partial f_3}{\partial x_2} & 0 & 0 \\ \dfrac{\partial f_4}{\partial x_1} & \dfrac{\partial f_4}{\partial x_2} & 0 & 0 \end{pmatrix}\right|_{\left(x_1, x_2, \dot{x}_1, \dot{x}_2\right) = \left(A_1, A_2, 0, 0\right)}.$$

Let us compute the partial derivatives

$$\frac{\partial f_3}{\partial x_1} = -\frac{2(\alpha+1)}{\left(x_1 - x_2\right)^{\alpha+2}} - \beta x_1^{\beta-1}, \quad \frac{\partial f_3}{\partial x_2} = \frac{2(\alpha+1)}{\left(x_1 - x_2\right)^{\alpha+2}}, \quad \frac{\partial f_4}{\partial x_1} = \frac{2(\alpha+1)}{\left(x_2 - x_1\right)^{\alpha+2}}, \quad \frac{\partial f_4}{\partial x_2} = -\frac{2(\alpha+1)}{\left(x_2 - x_1\right)^{\alpha+2}} - \beta x_2^{\beta-1}.$$

At the equilibrium point, one has $x_1 = A_1$, $x_2 = -A_1$, $x_1 - x_2 = 2A_1$, then the Jacobian matrix at the equilibrium point reads

$$\boldsymbol{J} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \zeta & 2(\alpha+1)\left(2A_1\right)^{-\alpha-2} & 0 & 0 \\ 2(\alpha+1)\left(-2A_1\right)^{-\alpha-2} & \eta & 0 & 0 \end{pmatrix},$$

where

$$\zeta := -2(\alpha+1)\left(2A_1\right)^{-\alpha-2} - \beta A_1^{\beta-1}, \qquad \eta := -2(\alpha+1)\left(-2A_1\right)^{-\alpha-2} - \beta\left(-A_1\right)^{\beta-1}.$$

Notice that for even α and odd β, $\eta = \zeta$, however, for odd α and even β, one has $\eta = -\zeta$.

To find the eigenvalues of Jacobian matrix $\boldsymbol{J}$, it suffices to calculate the eigenvalues of

$$\boldsymbol{M} := \begin{pmatrix} \zeta & 2(\alpha+1)\left(2A_1\right)^{-\alpha-2} \\ 2(\alpha+1)\left(-2A_1\right)^{-\alpha-2} & \eta \end{pmatrix}.$$

It can be verified as follows

$$0 = \det(\lambda \boldsymbol{I} - \boldsymbol{M}) = \begin{cases} (\lambda - \zeta)^2 - 4(\alpha+1)^2\left(2A_1\right)^{-2\alpha-4}, & \text{for even } \alpha \text{ and odd } \beta, \\ \lambda^2 - \zeta^2 + 4(\alpha+1)^2\left(2A_1\right)^{-2\alpha-4}, & \text{for odd } \alpha \text{ and even } \beta. \end{cases} \tag{11}$$

Therefore, the eigenvalues of Jacobian matrix $\boldsymbol{J}$ are square root of λ in (11).

For even α and odd β, the two eigenvalues of (11) are

$$\lambda_{\pm} := \zeta \pm 2(\alpha + 1)(2A_1)^{-\alpha - 2} < 0,$$

then the eigenvalues of Jacobian matrix $\boldsymbol{J}$ are $\pm \mathrm{i}\sqrt{-\lambda_{\pm}}$, which reveals that the constant solutions (10) are linearly stable.

For odd α and even β, the two eigenvalues of (11) are

$$\lambda_{\pm} := \pm\sqrt{\zeta^2 - 4(\alpha + 1)^2(2A_1)^{-2\alpha - 4}},$$

one sees that $\lambda_+ > 0$, then the Jacobian matrix $\boldsymbol{J}$ possesses one positive eigenvalue $\sqrt{\lambda_+}$, which reveals that the constant solutions (10) are linearly unstable. The proof is completed.

4 Blow up

This Section is devoted to the proof of Theorem 4 and Corollary 2, we investigate blow up behaviors of (PCM) for the case $\alpha = 2$, even β and $\gamma = -1$.

Firstly, we give the proof of Theorem 4 by employing the virial identity.

Proof We denote $V(t) = x_1^2 + x_2^2$, then one has

$$V''(t) = 2\left(\dot{x}_1^2 + \dot{x}_2^2 + x_1\ddot{x}_1 + x_2\ddot{x}_2\right).$$

Substituting the equations of motion $\ddot{x}_j = 2\left(x_j - x_k\right)^{-\alpha - 1} - x_j^{\beta}, j \neq k$, we obtain

$$V''(t) = 2\left(\dot{x}_1^2 + \dot{x}_2^2 + 2\left(x_1 - x_2\right)^{-2} - x_1^{\beta + 1} - x_2^{\beta + 1}\right).$$

By assumption $E(0) < -\varepsilon < 0$, which reveals that

$$\varepsilon \leqslant \dot{x}_1^2 + \dot{x}_2^2 + 2\left(x_1 - x_2\right)^{-2} + \varepsilon < -\frac{2}{\beta + 1}\left(x_1^{\beta + 1} + x_2^{\beta + 1}\right),$$

then one has

$$V''(t) = 2\left(\dot{x}_1^2 + \dot{x}_2^2 + 2\left(x_1 - x_2\right)^{-2} - x_1^{\beta + 1} - x_2^{\beta + 1}\right) \geqslant -2\left(x_1^{\beta + 1} + x_2^{\beta + 1}\right) > (\beta + 1)\varepsilon > 0.$$

Since for all $t, s > 0$,

$$V(t) = \int_0^t \int_0^s V''(\tau)\,\mathrm{d}\tau\,\mathrm{d}s > \frac{(\beta + 1)\varepsilon}{2}t^2.$$

The blowup occurs within $t \to +\infty$ and $\lim\limits_{t \to +\infty}\left(\left|x_1(t)\right| + \left|x_2(t)\right|\right) = +\infty$. This completes the proof.

Remark 3 The assumption $E(0) < -\varepsilon < 0$ can be achieved if we choose, for example, $x_1(0) \sim 0, x_2(0) << -1$ and then $x_1(0) - x_2(0)$ is large, the potential energy term $\left(x_1 - x_2\right)^{-2}$ is small and dominated by the term $-\frac{1}{\beta + 1}\left(x_1^{\beta + 1} + x_2^{\beta + 1}\right)$.

Proof of Corollary 2 It is easy to see that the total energy $E(t)$ is conserved for all $t \in \mathbb{R}$, where $E(t)$ is:

$$E(t) = E\left(x_1, x_2\right) = \frac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) - \left(x_1 - x_2\right)^{-2}.$$

Let $V(t) = x_1^2 + x_2^2$, then one has

$$V''(t) = 2\left(\dot{x}_1^2 + \dot{x}_2^2 - 2\left(x_1 - x_2\right)^{-2}\right) = 4E\left(x_1, x_2\right) = 4E(0) < 0. \tag{12}$$

Integrating (12) twice with respect to time yields

$$V(t) = 2E(0)t^2 + mt + n,$$

where m and n are constants. Since $E(0) < 0$, the parabola opens downward, then $V(t) > 0$ implies that

$$t < T_{max} = \frac{-m + \sqrt{m^2 - 8E(0)n}}{4E(0)}.$$

As $t \to T_{max}$, we have $V(t) = x_1^2 + x_2^2 \to 0^+$, which infers that $x_1, x_2 \to 0$ and the potential energy $2(x_1 - x_2)^{-2} \to +\infty$. Therefore,

$$\lim_{t \to T_{max}} \left(|\dot{x}_1(t)| + |\dot{x}_2(t)| \right) = +\infty.$$

The proof is completed.

5 Numerical simulation

In this Section, we employ the fourth-order Runge-Kutta method (RK4) to simulate the dynamics of solutions of (PCM), with a time step of $\Delta t = 0.01$ and integration interval $t \in [0,10]$. Notice that if $x_1 \approx x_2$, the denominator will approach zero, which leads to numerical explosions. A small perturbation $\epsilon = 10^{-8}$ is added to the denominator, we modifying it to $|x_1 - x_2| + \epsilon$. For α large, the interaction force decays rapidly. An implicit method (Radau integrator) is used with adaptive step size control (rtol=1e-6, atol=1e-9). For the first case of (PCM) with $\beta = 2$ and $\gamma = -1$, we study the following two scenarios:

Scenario 1 (Fixed initial values, varying α**)**: Observing the effect of different $\alpha \in \{1, 2, 4, 10, 50, 100\}$ over the solutions of (PCM). Here if x_1 and x_2 are fixed, we assume that the initial velocities $\dot{x}_1$ and $\dot{x}_2$ are zero.

Scenario 2 (Fixed α**, varying initial data)**: we compare the dynamics of solutions under different initial conditions.

5.1 Parameters

(i) Time step $\Delta t = 0.01$, integration duration $T = 10$.

(ii) Singularity perturbation $\epsilon = 10^{-8}$, state threshold $|x_j| < 10$ for $j = 1,2$.

(iii) Initial value combinations:

① Scenario 1: Four sets of initial values:

$a = 1, b = 2, c = 0, d = 0$;　　$a = 1, b = 2.5, c = 0, d = 0$;

$a = 1.5, b = 2, c = 0, d = 0$;　　$a = 1.5, b = 2.5, c = 0, d = 0$.

② Scenario 2: Six types of α. $\alpha \in \{1, 2, 4, 10, 50, 100\}$.

5.2 Experimental results

Case 1: Fixed initial values, varying α, see Fig. 1.

It can be seen that the system exhibits significant convergence characteristics. When α varies from 1 to 100, regardless of its value, the solutions eventually stabilize. Taking the initial value $(1, 2, 0, 0)$ as an example, all trajectory curves reach stability as α increases from 1 to 100, which confirms the stability mechanism of the system. Notably, the convergence speed is positively correlated with α: when $\alpha = 1$, particles take a longer time to slowly approach stable positions; whereas for $\alpha \geqslant 10$, convergence accelerates, especially when α reaches 50 or 100, with particles quickly locking into stable values.

Differences in initial conditions significantly impact the dynamics of the system. Comparing the trajectories of initial values $(1, 2, 0, 0)$ and $(1.5, 2.5, 0, 0)$, there are noticeable deviations in early evolution paths, but over time, their stable positions gradually converge. This phenomenon indicates that α not only regulates convergence speed but also determines stable positions. As α increases from 1 to 100, the particles possess a monotonic upward trend. For instance, $\alpha = 1$ corresponds to a lower equilibrium point, while $\alpha = 50$ results in a significantly higher balance. Particularly, when α exceeds a critical value (approximately above 50), the associated nonlinear response weakens, and particle motion exhibits quasi-linear characteristics, ultimately stabilizing at some higher points.

Case 2: Fixed α**, varying initial data,** see Fig. 2.

The system demonstrates clear convergence characteristics in long tme evolution when α is fixed. Regardless of initial positions and velocities, particles eventually stabilize at specific values. For example, with

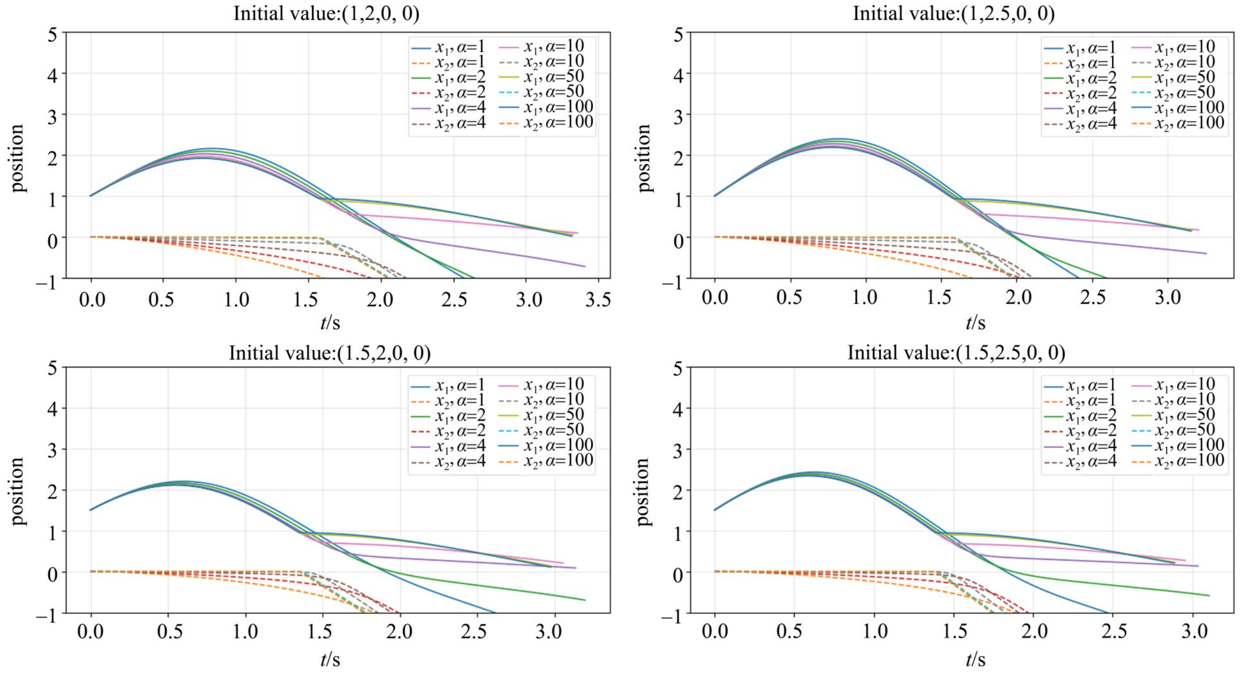

Fig. 1 Trajectory of solutions for (PCM) with fixed initial data and varying α

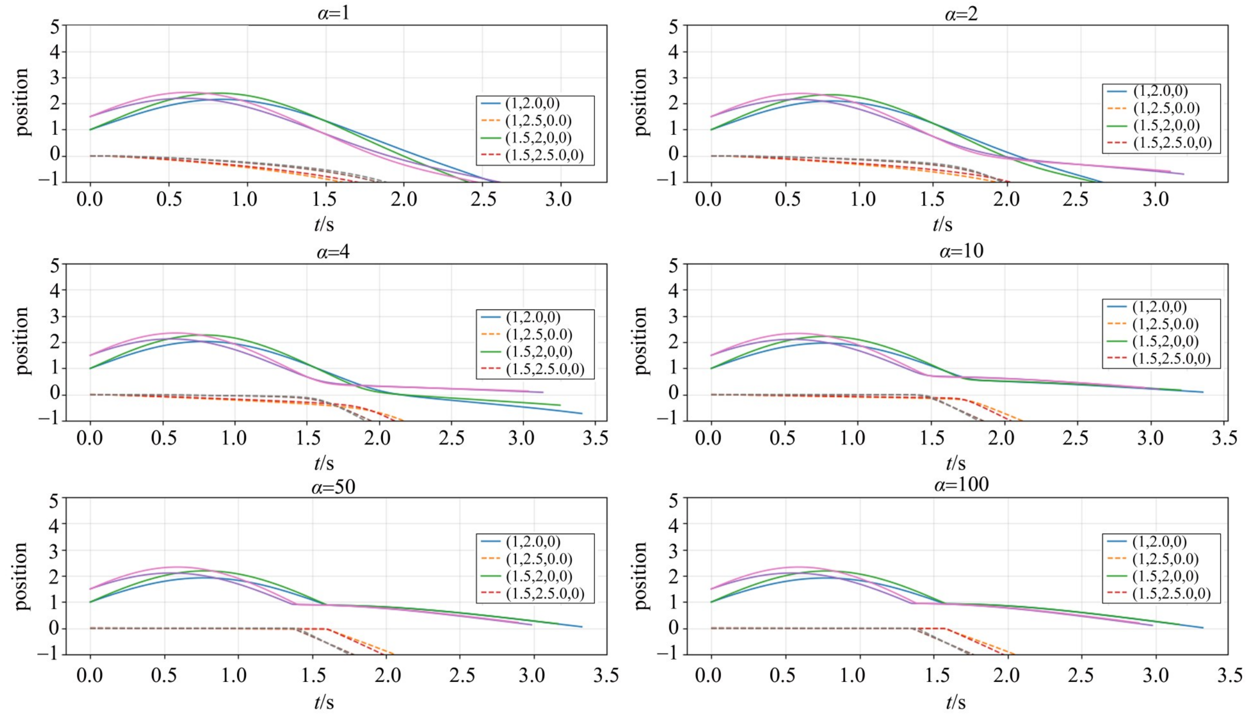

Fig. 2 Trajectory of solutions for (PCM) with fixed α and varying initial values

$\alpha = 1$, multiple different initial values (such as $(1,2,0,0)$, $(1.5,2,0,0)$, etc.) all converge to the same stable point. Notably, the value of α significantly affects the convergence process: Larger α (e.g., $\alpha \geqslant 10$) leads to faster stabilization and higher stable positions. For instance, $\alpha = 1$ results in a lower stable point with slower convergence, while $\alpha = 50$ sees a conspicuous higher stable point and rapid stabilization.

Differences in initial conditions mainly influence the dynamical behavior during early evolution. Taking the comparison of initial values $(1,2,0,0)$ and $(1.5,2.5,0,0)$ as examples, their paths differ significantly in the early stages, but over time, the trajectories gradually align, with negligible differences in final stable values. This phenomenon reveals the initial perturbations are progressively dissipated during evolution. Further analysis indicates that the sensitivity of α requires special attention: its numerical changes not only alter the stable position and convergence rate but may also trigger shifts, which needs to be finely tuned in practical applications to achieve desired behavior.

We also conduct numerical simulations for the solutions of (PCM) with $\alpha = 2, \beta = 2$ and $\gamma = 1$ and negative energies, it shows that the solutions must exhibit blow up. We consider the following two specific initial conditions as examples, see Fig. 3 .

Initial condition 1: $x_1(0) = 5.0, x_2(0) = 6.0, \dot{x}_1(0) = 0, \dot{x}_2(0) = 0$ with $\alpha = 2$. Here, $|x_1(0) - x_2(0)| = 1.0 \geqslant \delta = 0.5$ and the initial energy $E(0) \approx -112.7 < 0$.

Initial condition 2: $x_1(0) = 8.0, x_2(0) = 9.5, \dot{x}_1(0) = 0, \dot{x}_2(0) = 0$ with $\alpha = 2$. Here, $|x_1(0) - x_2(0)| = 1.5 \geqslant \delta = 0.5$ and the initial energy $E(0) \approx -445.8 < 0$.

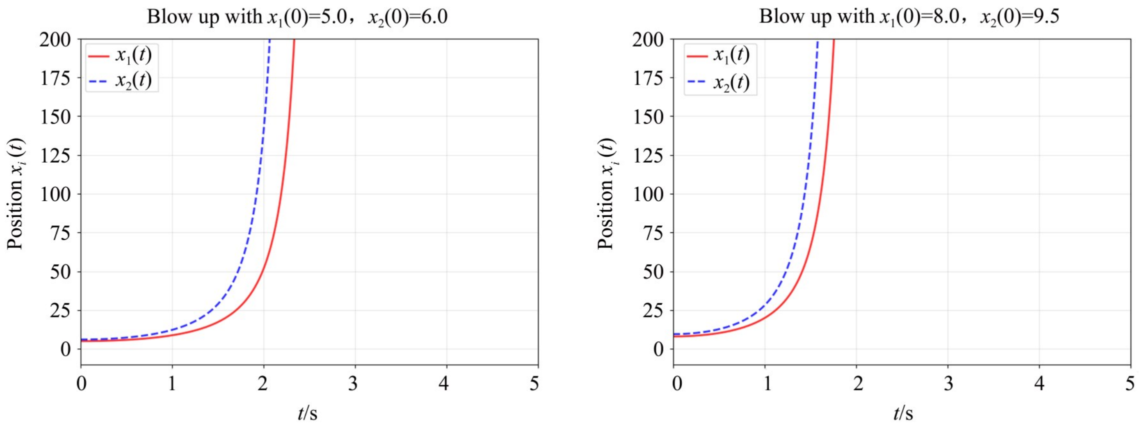

Fig. 3 Verification of the burstability of the solution

The numerical simulations reveal that the solutions exhibit blow up behavior, which diverge rapidly at $t \approx 2.5$ and $t \approx 1.8$ for the respective initial conditions. This explosive growth stems from the dominant influence of the negative cubic potential term $-\frac{1}{3}(x_1^3 + x_2^3)$.

Acknowledgements: Z. Wang is indebted to Prof. Robert Pego for stimulating discussions.

References:

ABANOV A G, BETTELHEIM E, WIEGMANN P, 2009. Integrable hydrodynamics of Calogero-Sutherland model: Bidirectional Benjamin-Ono equation[J]. J Phys A: Math Theor, 42(13): 135201.

ABANOV A G, GROMOV A, KULKARNI M, 2011. Soliton solutions of a Calogero model in a harmonic potential[J]. J Phys A: Math Theor, 44(29): 295203.

CALOGERO F, 1971. Solution of the one-dimensional *N*-body problems with quadratic and/or inversely quadratic pair potentials [J]. J Math Phys, 12(3): 419-436.

CASE K M, 1978. The *N*-soliton solution of the Benjamin-Ono equation[J]. Proc Natl Acad Sci, 75(8): 3562-3563.

ENGLISH J M, PEGO R L, 2005. On the solitary wave pulse in a chain of beads[J]. Proc Amer Math Soc, 133(6): 1763-1768.

FLACH S, GORBACH A V, 2008. Discrete breathers with dissipation[M]//AKHMEDIEV N, et al. Dissipative solitons: From optics to biology and medicine, Lect Notes Phys, 751, Berlin: Springer.

INGIMARSON B, PEGO R L, 2024a. On long waves and solitons in particle lattices with forces of infinite range[J]. SIAM J Appl

Math, 84 (3): 808–830.

INGIMARSON B, PEGO R L, 2024b. Existence of solitary waves in particle lattices with power-law forces[J]. Nonlinearity, 37 (12): 125016.

MOLERÓN M, LEONARD A, DARAIO C, 2014. Solitary waves in a chain of repelling magnets[J]. J Appl Phys, 115(18): 184901.

MOSER J, 1975. Three integrable Hamiltonian systems connected with isospectral deformations[J]. Adv Math, 16(2): 197–220.

OLSHANETSKY M A, PERELOMOV A M, 1981. Classical integrable finite-dimensional systems related to Lie algebras[J]. Phys Rep, 71(5): 313–400.

PHILIP A R, 2019. Soliton solutions to Calogero-Moser systems[D]. Stockholm: Royal Institute of Technology.

POLYCHRONAKOS A P, 1995. Waves and solitons in the continuum limit of the Calogero-Sutherland model[J]. Phys Rev Lett, 74(26): 5153–5157.

STONE M, ANDUAGA I, XING L, 2008. The classical hydrodynamics of the Calogero-Sutherland model[J]. J Phys A: Math Theor, 41(27): 275401.

SUTHERLAND B, 1971. Exact results for a quantum many-body problem in one dimension[J]. Phys Rev A, 4(5): 2019–2021.

WOJCIECHOWSKI S, 1982. The analogue of the Bäcklund transformation for integrable many-body systems[J]. J Phys A: Math Gen, 15(12): L653–L657.

受扰Calogero-Moser系统的动力学行为:适定性、稳定性和爆破

刘千乐，王忠，朱伟鹏

佛山大学数学学院，广东 佛山 528000

摘要：研究了一类受幂律型外势场调制的不可积双粒子Calogero-Moser系统．在粒子满足严格初始分离条件下，建立了柯西问题的局部适定性．对于适当的初始配置，局部解可通过能量守恒全局延拓；反之，负能量条件将诱发有限或无限时间爆破．当系统能量处于临界阈值时，分析了稳态解的线性(不)稳定性．数值模拟采用带自适应步长控制的四阶龙格-库塔算法，模拟结果表明：轨迹演化依幂指数α和初始条件不同，或收敛至稳态或呈现爆破．增大α值会加快收敛速度并抑制振荡动力学行为，促使系统从周期性运动向静态平衡转变．

关键词：Calogero-Moser系统；适定性；爆破；稳定性

（责任编辑　冯兆永）