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具有垂直传染的病毒变异传染病模型分析*

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摘　要：本文以COVID-19为背景，在年龄结构的传染病问题中，首次考虑三类人群同时具有传染性的情况，建立一类具有垂直传染和潜伏期的病毒变异传染病模型．首先利用微分方程定性理论证得模型非负解的存在唯一性，并给出疾病流行的基本再生数．接着，借助微分方程稳定性理论，证明得到无病平衡点存在与稳定的充分条件．最后证明得到地方病平衡点的存在性．

关键词：垂直传染；病毒变异；传染病模型

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Analysis of a virus mutation epidemic model with vertical transmission

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Abstract： Against the backdrop of COVID-19，we consider the scenario where three groups of people are simultaneously infectious in the age-structured infectious disease problem，it was established that a virus mutation epidemic model with vertical transmission and latency period. Firstly，the existence and uniqueness of non-negative solutions are proved，and the basic regeneration number of the model was obtained by using the qualitative theory of ordinary differential equation. Secondly，the sufficient conditions for the existence and stability of the disease-free equilibrium point are proved by using the stability theory of ordinary differential equationv. Finally，the existence of the endemic equilibrium point is demonstrated.

Key words： vertical transmission；virus mutation；epidemic model

纵观历史，从鼠疫、天花、甲型HINI流感到COVID-19，传染病的暴发给人类健康、经济和社会秩序造成了严重的影响．由于实验条件难以完全模拟真实场景，通过理论构建数学模型模拟疾病传播的动力学特征成为了解传染病传播机制的主要途径．Kermack et al.（1991a）为研究黑死病的流行规律，利用动力学方法建立了SIR仓室模型，Kermack et al.（1991b）提出了著名的传染病传播阈值理论．此后，诸多学者对不同仓室的传染病模型展开研究（Li et al.，2009；Kaddar et al.，2017；Liu et al.，2018；Yang et al.，2019），并取得了显著的成果．然而随着疾病的发展，病毒在传播过程中受到多种因素的影响可能会发生变异，从而导致疾病失控．Zhu et al.（2025）利用EKF对疫苗接种和病毒变异的COVID-19传染病模型进行了预测．高文哲（2023）探讨

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了具有病毒变异的传染病模型，但只考虑了其中一种病毒具有传染性的情况．Alzamora et al.（2020）揭示了COVID-19可能存在垂直传染的可能，陈姗姗等（2020）与Pandit et al.（2022）的研究进一步证实了孕产妇COVID-19患者有较低风险会发生垂直传染．马怡婷（2023）对潜伏期具有传染性的病毒变异传染病模型进行了研究．Abid et al.（2012）与Naim et al.（2021）都考虑了潜伏期和染病期具有传染性的模型，分析了系统解的稳定性．基于上述研究，本文在年龄结构的传染病问题中，同时考虑潜在垂直传染风险和病毒变异的情况，提出一类具有垂直传染和潜伏期的$MSEI_1I_2R$病毒变异传染病模型，假设潜伏者和病毒变异前后染病者同时具有传染性，研究疾病流行的阈值与模型平衡点的存在与稳定条件，为疾病的传播与防控提供一定的理论依据．

1　模型建立与基本假设

本文首次在同一模型中考虑三类人群都具有传染性的情况，建立一类具有垂直传染和潜伏期的病毒变异传染病模型．假设所有染病者分为潜伏者，变异前患者和变异后患者，部分变异前患者未能治愈而发展成变异后患者，且这三类人群都具有传染性，他们所生育的下一代一部分成为潜伏者，另一部分短期内拥有自然免疫力，称为被动免疫者，被动免疫力丧失后成为易感染者，于是建立如下模型：

$$\begin{cases}\dfrac{\partial M(a,t)}{\partial a}+\dfrac{\partial M(a,t)}{\partial t}=-\delta(a)M(a,t)-\mu(a)M(a,t),\\ \dfrac{\partial S(a,t)}{\partial a}+\dfrac{\partial S(a,t)}{\partial t}=\delta(a)M(a,t)-\lambda(a,t)S(a,t)-\mu(a)S(a,t),\\ \dfrac{\partial E(a,t)}{\partial a}+\dfrac{\partial E(a,t)}{\partial t}=\lambda(a,t)S(a,t)-\left[\phi(a)E(a,t)+\gamma_1(a)+\mu(a)\right]E(a,t),\\ \dfrac{\partial I_1(a,t)}{\partial a}+\dfrac{\partial I_1(a,t)}{\partial t}=\phi(a)E(a,t)-\left[\varepsilon(a)+\gamma_2(a)+\mu(a)\right]I_1(a,t),\\ \dfrac{\partial I_2(a,t)}{\partial a}+\dfrac{\partial I_2(a,t)}{\partial t}=\varepsilon(a)I_1(a,t)-\left[\gamma_3(a)+\mu(a)\right]I_2(a,t),\\ \dfrac{\partial R(a,t)}{\partial a}+\dfrac{\partial R(a,t)}{\partial t}=\gamma_1(a)E(a,t)+\gamma_2(a)I_1(a,t)+\gamma_3(a)I_2(a,t)-\mu(a)R(a,t),\end{cases}\tag{1}$$

其中$(a,t)\in Q=[0,T]\times[0,a^+]$，$M(a,t),S(a,t),\ E(a,t),\ I_1(a,t),\ \ I_2(a,t),\ R(a,t)$分别表示$t$时刻年龄为$a$的被动免疫者、易感染者、潜伏者、变异前患者、变异后患者和恢复者的密度函数，$\mu(a)$表示年龄依赖自然死亡率，$\delta(a)^{-1}$，$\phi(a)^{-1}$，$\varepsilon(a)^{-1}$分别表示被动免疫周期、潜伏周期和病毒变异周期，$\gamma_1(a),\ \gamma_2(a),\ \gamma_3(a)$分别表示潜伏者、变异前患者和变异后患者的年龄依赖恢复周期，$b(a)$表示年龄依赖出生率．不考虑因病死亡率，$k_1,\ k_2,\ k_3$分别表示$E(a,t),\ I_1(a,t),\ I_2(a,t)$的垂直传染率，传染率函数为

$$\lambda(a,t)=k(a)\int_0^{a^+}\left\{\beta_1(a)E(a,t)+\beta_2(a)\left[I_1(a,t)+I_2(a,t)\right]\right\}\mathrm{d}a,$$

这里，$k(a)$表示年龄依赖接触率，$\beta_1(a)$表示$E(a,t)$的年龄依赖传染率，$\beta_2(a)$表示$I_1(a,t),I_2(a,t)$的年龄依赖传染率．

设初值和边值条件为

$$\begin{cases}M(0,t)=\displaystyle\int_0^{a^+}b(a)\left[M(a,t)+(1-k_1)E(a,t)+(1-k_2)I_1(a,t)+(1-k_3)I_2(a,t)+R(a,t)\right]\mathrm{d}a,\\ S(0,t)=\displaystyle\int_0^{a^+}b(a)S(a,t)\mathrm{d}a,\\ E(0,t)=\displaystyle\int_0^{a^+}b(a)\left[k_1E(a,t)+k_2I_1(a,t)+k_3I_2(a,t)\right]\mathrm{d}a,\\ I_1(0,t)=I_2(0,t)=R(0,t)=0,\qquad M(a,0)=M_0(a),\quad S(a,0)=S_0(a),\\ E(a,0)=E_0(a),\quad I_1(a,0)=I_{10}(a),\quad I_2(a,0)=I_{20}(a),\quad R(a,0)=R_0(a),\end{cases}\tag{2}$$

其中$t\in[0,T]$，$a\in[0,a^+]$．

对模型作如下基本假设：

（A1）$b(a),\ \delta(a),\ \phi(a),\varepsilon(a),\ \gamma_1(a),\ \gamma_2(a),\ \gamma_3(a),k(a),\beta_1(a),\beta_2(a)\in L^{\infty}(0,a^+)$，且均非负．

（A2）$M_0(a),S_0(a),E_0(a),\ I_{10}(a),\ I_{20}(a),\ R_0(a)\in L^1(0,a^+)$，且均为非负数．

（A3）$\mu(a)\in L^{\infty}_{loc}(0,a^+),\ \mu(a)\geqslant 0$，且$\int_0^{\infty}\mu(a)\mathrm{d}a=+\infty$.

2 解的存在唯一性

定义1 若$(M,S,E,\ I_1,\ I_2,\ R)\in\left(L^{\infty}\left(0,T;\ L^1(0,a^+)\right)\right)^6$在每条特征线$a-t=c$上都绝对连续，$(a,t)\in Q,\ c\in\mathbf{R}$且满足模型（1）与对任意$t\in[0,T],a\in[0,a^+]$有

$$\begin{cases}\lim\limits_{\varepsilon\to0^+}M(\varepsilon,t+\varepsilon)=\int_0^{a^+}b(a)\left[M(a,t)+(1-k_1)E(a,t)+(1-k_2)I_1(a,t)+(1-k_3)I_2(a,t)+R(a,t)\right]\mathrm{d}a,\\ \lim\limits_{\varepsilon\to0^+}S(\varepsilon,t+\varepsilon)=\int_0^{a^+}b(a)S(a,t)\mathrm{d}a,\qquad \lim\limits_{\varepsilon\to0^+}E(\varepsilon,t+\varepsilon)=\int_0^{a^+}b(a)\left[k_1E(a,t)+k_2I_1(a,t)+k_3I_2(a,t)\right]\mathrm{d}a,\\ \lim\limits_{\varepsilon\to0^+}I_1(\varepsilon,t+\varepsilon)=0,\qquad \lim\limits_{\varepsilon\to0^+}I_2(\varepsilon,t+\varepsilon)=0,\qquad \lim\limits_{\varepsilon\to0^+}R(\varepsilon,t+\varepsilon)=0,\\ \lim\limits_{\varepsilon\to0^+}M(a+\varepsilon,\varepsilon)=M_0(a),\qquad \lim\limits_{\varepsilon\to0^+}S(a+\varepsilon,\varepsilon)=S_0(a),\qquad \lim\limits_{\varepsilon\to0^+}E(a+\varepsilon,\varepsilon)=E_0(a),\\ \lim\limits_{\varepsilon\to0^+}I_1(a+\varepsilon,\varepsilon)=I_{10}(a),\qquad \lim\limits_{\varepsilon\to0^+}I_2(a+\varepsilon,\varepsilon)=I_{20}(a),\qquad \lim\limits_{\varepsilon\to0^+}R(a+\varepsilon,\varepsilon)=R_0(a),\end{cases}$$

则称(M,S,E,I_1,I_2,R)为系统（1）~（2）的解．

下面讨论解的存在性，令

$$L_1(M,S,E,I_1,I_2,R)=-\delta(a)M(a,t),\qquad L_2(M,S,E,I_1,I_2,R)=\delta(a)M(a,t)-\lambda(a,t)S(a,t),$$
$$L_3(M,S,E,I_1,I_2,R)=\lambda(a,t)S(a,t)-\left[\phi(a)+\gamma_1(a)\right]E(a,t),$$
$$L_4(M,S,E,I_1,I_2,R)=\phi(a)E(a,t)-\left[\varepsilon(a)+\gamma_2(a)\right]I_1(a,t),$$
$$L_5(M,S,E,I_1,I_2,R)=\varepsilon(a)I_1(a,t)-\gamma_3(a)I_2(a,t),$$
$$L_6(M,S,E,I_1,I_2,R)=\gamma_1(a)E(a,t)+\gamma_2(a)I_1(a,t)+\gamma_3(a)I_2(a,t).$$

则模型（1）简化为

$$\begin{cases}\dfrac{\partial M(a,t)}{\partial a}+\dfrac{\partial M(a,t)}{\partial t}=L_1(M,S,E,I_1,I_2,R)-\mu(a)M(a,t),\\ \dfrac{\partial S(a,t)}{\partial a}+\dfrac{\partial S(a,t)}{\partial t}=L_2(M,S,E,I_1,I_2,R)-\mu(a)S(a,t),\\ \dfrac{\partial E(a,t)}{\partial a}+\dfrac{\partial E(a,t)}{\partial t}=L_3(M,S,E,I_1,I_2,R)-\mu(a)E(a,t),\\ \dfrac{\partial I_1(a,t)}{\partial a}+\dfrac{\partial I_1(a,t)}{\partial t}=L_4(M,S,E,I_1,I_2,R)-\mu(a)I_1(a,t),\\ \dfrac{\partial I_2(a,t)}{\partial a}+\dfrac{\partial I_2(a,t)}{\partial t}=L_5(M,S,E,I_1,I_2,R)-\mu(a)I_2(a,t),\\ \dfrac{\partial R(a,t)}{\partial a}+\dfrac{\partial R(a,t)}{\partial t}=L_6(M,S,E,I_1,I_2,R)-\mu(a)R(a,t).\end{cases}\tag{3}$$

用$x_i(i=1,2,\cdots,6)$表示M,S,E,I_1,I_2,R，定义解空间

$$X\coloneqq\left\{x=(x_1,x_2,x_3,x_4,x_5,x_6)\in\left(L^{\infty}\left(0,T;\ L^1(0,a^+)\right)\right)^6\,\middle|\,\sum_{i=1}^{6}\|x_i\|_{L^{\infty}\left(0,T;\ L^1(0,a^+)\right)}<+\infty,\ x_i\geqslant0,\ i=1,2,\cdots,6\right\}.$$

利用特征线法求解系统（2）~（3）得

$$x_i=\begin{cases}\mathrm{e}^{-\int_0^a\mu(\tau)\mathrm{d}\tau}x_i(0,t-a)+\int_0^a\mathrm{e}^{-\int_\sigma^a\mu(\tau)\mathrm{d}\tau}L_i(\sigma,\sigma+t-a)\mathrm{d}\sigma, & a<t,\\ \mathrm{e}^{-\int_0^t\mu(\tau-t+a)\mathrm{d}\tau}x_i(a-t,0)+\int_0^t\mathrm{e}^{-\int_0^t\mu(\tau-t+a)\mathrm{d}\tau}L_i(\sigma+a-t,\sigma)\mathrm{d}\sigma, & a>t.\end{cases}$$

令$P(a,t)=M(a,t)+S(a,t)+E(a,t)+I_1(a,t)+I_2(a,t)+R(a,t)$，由模型（1）得关于人口密度函数$P(a,t)$的系统：

$$\begin{cases}\dfrac{\partial P(a,t)}{\partial a}+\dfrac{\partial P(a,t)}{\partial t}=-\mu(a)P(a,t), & (a,t)\in Q,\\ P(0,t)=\displaystyle\int_0^{+\infty}b(a)P(a,t)\mathrm{d}a, & (a,t)\in Q,\\ P(a,0)=P_0(a)=M_0(a)+S_0(a)+E_0(a)+I_{10}(a)+I_{20}(a)+R_0(a), & (a,t)\in Q.\end{cases}$$

这是标准的McKendrick-Von Forester人口发展方程.由Anita(2000)知,在$(0,T)$上几乎处处满足$\|P(\cdot,t)\|_{L^1(0,a^+)}\leqslant C_T$,其中$C_T=\max\left\{\|P_0\|_{L^1(0,a^+)},\|F\|_{L^\infty(0,T)}\mathrm{e}^{T\|b\|_{L^\infty(0,a^+)}}\right\}$,$F(t)=\int_t^{a^+}b(a+t)P_0(a)\mathrm{e}^{-\int_0^t\mu(\tau+a)\mathrm{d}\tau}\mathrm{d}a$.由高文哲(2023)可证得系统非负解的存在唯一性.

定理1　在条件(A1)~(A3)下系统(1)~(2)在空间X中存在唯一的非负解(M,S,E,I_1,I_2,R).

3　基本再生数与无病平衡点的稳定性

本节给出基本再生数R_0的表达式,并证明无病平衡点E_0的渐近稳定性.

3.1　基本再生数

设人口净再生数为1,即$\int_0^{a^+}b(a)\mathrm{e}^{-\int_0^a\mu(\sigma)d\sigma}\mathrm{d}a=1$,则总人口密度函数达到稳定状态,设$t\to+\infty$时年龄为$a$的人口密度为$P_\infty(a)$,$b_0$为新生婴儿的出生率,则人口密度满足$P(a,t)=P_\infty(a)=P(a,t)=b_0\mathrm{e}^{-\int_0^a\mu(\sigma)\mathrm{d}\sigma}$,则$b_0=\int_0^{a^+}P_\infty(a)\mathrm{d}a\Big/\int_0^{a^+}\mathrm{e}^{-\int_0^a\mu(\sigma)\mathrm{d}\sigma}\mathrm{d}a$.

对模型(1)做归一化变换,令

$$m(a,t)=\frac{M(a,t)}{P_\infty(a)},\qquad s(a,t)=\frac{S(a,t)}{P_\infty(a)},\qquad e(a,t)=\frac{E(a,t)}{P_\infty(a)},$$
$$i_1(a,t)=\frac{I_1(a,t)}{P_\infty(a)},\qquad i_2(a,t)=\frac{I_2(a,t)}{P_\infty(a)},\qquad r(a,t)=\frac{R(a,t)}{P_\infty(a)}.$$

则$\tilde{\lambda}(a,t)=k(a)\int_0^{a^+}P_\infty(a)\left\{\beta_1(a)e(a,t)+\beta_2(a)\left[i_1(a,t)+i_2(a,t)\right]\right\}\mathrm{d}a=k(a)V(t)$,其中

$$V(t)=\int_0^{a^+}P_\infty(a)\left\{\beta_1(a)e(a,t)+\beta_2(a)\left[i_1(a,t)+i_2(a,t)\right]\right\}\mathrm{d}a.$$

系统(1)~(2)转化为

$$\begin{cases}\dfrac{\partial m(a,t)}{\partial a}+\dfrac{\partial m(a,t)}{\partial t}=-\delta(a)m(a,t),\\ \dfrac{\partial s(a,t)}{\partial a}+\dfrac{\partial s(a,t)}{\partial t}=\delta(a)m(a,t)-\tilde{\lambda}(a,t)s(a,t),\\ \dfrac{\partial e(a,t)}{\partial a}+\dfrac{\partial e(a,t)}{\partial t}=\tilde{\lambda}(a,t)s(a,t)-\left[\phi(a)+\gamma_1(a)\right]e(a,t),\\ \dfrac{\partial i_1(a,t)}{\partial a}+\dfrac{\partial i_1(a,t)}{\partial t}=\phi(a)e(a,t)-\left[\varepsilon(a)+\gamma_2(a)\right]i_1(a,t),\\ \dfrac{\partial i_2(a,t)}{\partial a}+\dfrac{\partial i_2(a,t)}{\partial t}=\varepsilon(a)i_1(a,t)-\gamma_3(a)i_2(a,t),\\ \dfrac{\partial r(a,t)}{\partial a}+\dfrac{\partial r(a,t)}{\partial t}=\gamma_1(a)e(a,t)+\gamma_2(a)i_1(a,t)+\gamma_3(a)i_2(a,t),\\ m(0,t)=\displaystyle\int_0^{a^+}b(a)\left[m(a,t)+(1-k_1)e(a,t)+(1-k_2)i_1(a,t)+(1-k_3)i_2(a,t)+r(a,t)\right]\mathrm{e}^{-\int_0^a\mu(\sigma)\mathrm{d}\sigma}\mathrm{d}a,\\ s(0,t)=\displaystyle\int_0^{a^+}b(a)s(a,t)\mathrm{e}^{-\int_0^a\mu(\sigma)\mathrm{d}\sigma}\mathrm{d}a,\quad e(0,t)=\displaystyle\int_0^{a^+}b(a)\left\{k_1e(a,t)+k_2i_1(a,t)+k_3i_2(a,t)\right\}\mathrm{e}^{-\int_0^a\mu(\sigma)\mathrm{d}\sigma}\mathrm{d}a,\\ i_1(0,t)=i_2(0,t)=r(0,t)=0,\quad m(a,0)=m_0(a),\quad s(a,0)=s_0(a),\quad e(a,0)=e_0(a),\quad i_1(a,0)=i_{10}(a),\\ i_2(a,0)=i_{20}(a),\quad r(a,0)=r_0(a),\quad m(a,t)+s(a,t)+e(a,t)+i_1(a,t)+i_2(a,t)+r(a,t)=1.\end{cases}\tag{4}$$

模型(4)的平衡解满足

$$\begin{cases}
\dfrac{\mathrm{d}m(a)}{\mathrm{d}t}=-\delta(a)m(a),\\
\dfrac{\mathrm{d}s(a)}{\mathrm{d}t}=\delta(a)m(a)-\tilde{\lambda}(a)s(a),\\
\dfrac{\mathrm{d}e(a)}{\mathrm{d}t}=\tilde{\lambda}(a)s(a)-\left[\phi(a)+\gamma_1(a)\right]e(a),\\
\dfrac{\mathrm{d}i_1(a)}{\mathrm{d}t}=\phi(a)e(a)-\left[\varepsilon(a)+\gamma_2(a)\right]i_1(a),\\
\dfrac{\mathrm{d}i_2(a)}{\mathrm{d}t}=\varepsilon(a)i_1(a)-\gamma_3(a)i_2(a),\\
\dfrac{\mathrm{d}r(a)}{\mathrm{d}t}=\gamma_1(a)e(a)+\gamma_2(a)i_1(a)+\gamma_3(a)i_2(a),\\
m(0)=\displaystyle\int_0^{a^+}b(a)\left[m(a)+\left(1-k_1\right)e(a)+\left(1-k_2\right)i_1(a)+\left(1-k_3\right)i_2(a)+r(a)\right]\mathrm{e}^{-\int_0^a\mu(\sigma)\mathrm{d}\sigma}\mathrm{d}a,\\
s(0)=\displaystyle\int_0^{a^+}b(a)s(a)\mathrm{e}^{-\int_0^a\mu(\sigma)\mathrm{d}\sigma}\mathrm{d}a,\\
e(0)=\displaystyle\int_0^{a^+}b(a)\left[k_1e(a)+k_2i_1(a)+k_3i_2(a)\right]\mathrm{e}^{-\int_0^a\mu(\sigma)\mathrm{d}\sigma}\mathrm{d}a,\\
i_1(0)=i_2(0)=r(0)=0,m(a)+s(a)+e(a)+i_1(a)+i_2(a)+r(a)=1.\\
\tilde{\lambda}(a)=k(a)\displaystyle\int_0^{a^+}P_\infty(a)\left\{\beta_1(a)e(a)+\beta_2(a)\left[i_1(a)+i_2(a)\right]\right\}\mathrm{d}a=k(a)V_0.
\end{cases}\tag{5}$$

由式(5)解得

$$m(a)=m(0)\mathrm{e}^{-\int_0^a\delta(\sigma)\mathrm{d}\sigma},\qquad s(a)=s(0)\mathrm{e}^{-\int_0^a\tilde{\lambda}(\sigma)\mathrm{d}\sigma}+m(0)\int_0^a\delta(\xi)\mathrm{e}^{-\int_0^\xi\delta(\sigma)\mathrm{d}\sigma}\mathrm{e}^{-\int_\xi^a\tilde{\lambda}(\sigma)\mathrm{d}\sigma}\mathrm{d}\xi,$$

$$r(a)=\int_0^a\left[\gamma_1(\xi)e(\xi)+\gamma_2(\xi)i_1(\xi)+\gamma_3(\xi)i_2(\xi)\right]\mathrm{d}\xi.$$

当$V_0=0$时，$e(a)=0,\ i_1(a)=0,\ i_2(a)=0,\ r_0(a)=0$，则$s(0)=s(0)F_1\left(V_0\right)+m(0)F_2\left(V_0\right)$，其中

$$F_1\left(V_0\right)=\int_0^{a^+}b(a)\mathrm{e}^{-\int_0^a\mu(\sigma)\mathrm{d}\sigma}\mathrm{e}^{-\int_0^a k(\sigma)V_0\mathrm{d}\sigma}\mathrm{d}a,\qquad F_2\left(V_0\right)=\int_0^{a^+}\int_0^a b(a)\mathrm{e}^{-\int_0^a\mu(\sigma)\mathrm{d}\sigma}\delta(\xi)\mathrm{e}^{-\int_0^\xi\delta(\sigma)\mathrm{d}\sigma}\mathrm{e}^{-\int_\xi^a k(\sigma)V_0\mathrm{d}\sigma}\mathrm{d}\xi\,\mathrm{d}a\,.$$

当$V_0=0$时，$F_1\left(V_0\right)=\int_0^{a^+}b(a)\mathrm{e}^{-\int_0^a\mu(\sigma)\mathrm{d}\sigma}\mathrm{d}a=1$，所以$m(0)=0,\ s(0)=1$. 故无病平衡点为$E_0(0,1,0,0,0,0)$.

对模型(4)在E_0处进行线性化，考虑如下指数形式的解

$$m(a,t)=\bar{m}(a)\mathrm{e}^{wt},\qquad s(a,t)=1+\bar{s}(a)\mathrm{e}^{wt},\qquad e(a,t)=\bar{e}(a)\mathrm{e}^{wt},$$
$$i_1(a,t)=\bar{i}_1(a)\mathrm{e}^{wt},\qquad \bar{i}_2(a,t)=\bar{i}_2(a)\mathrm{e}^{wt},\qquad r(a,t)=\bar{r}(a)\mathrm{e}^{wt}.$$

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$$\begin{cases}
\dfrac{\mathrm{d}\bar{m}(a)}{\mathrm{d}a}+w\bar{m}(a)=-\delta(a)\bar{m}(a),\\
\dfrac{\mathrm{d}\bar{s}(a)}{\mathrm{d}a}+w\bar{s}(a)=\delta(a)\bar{m}(a)-\bar{\lambda}(a),\\
\dfrac{\mathrm{d}\bar{e}(a)}{\mathrm{d}a}+w\bar{e}(a)=\bar{\lambda}(a)-\left[\phi(a)+\gamma_1(a)\right]\bar{e}(a),\\
\dfrac{\mathrm{d}\bar{i}_1(a)}{\mathrm{d}a}+w\bar{i}_1(a)=\phi(a)\bar{e}(a)-\left[\varepsilon(a)+\gamma_2(a)\right]\bar{i}_1(a),\\
\dfrac{\mathrm{d}\bar{i}_2(a)}{\mathrm{d}a}+w\bar{i}_2(a)=\varepsilon(a)\bar{i}_1(a)-\gamma_3(a)\bar{i}_2(a),\\
\dfrac{\mathrm{d}\bar{r}(a)}{\mathrm{d}a}+w\bar{r}(a)=\gamma_1(a)\bar{e}(a)+\gamma_2(a)\bar{i}_1(a)+\gamma_3(a)\bar{i}_2(a),\\
\bar{m}(0)=\bar{s}(0)=\bar{e}(0)=\bar{i}_1(0)=\bar{i}_2(0)=\bar{r}(0)=0,
\end{cases}\tag{6}$$

其中

$$\bar{\lambda}(a) = k(a)\int_0^{a^+} P_\infty(a)\left\{\beta_1(a)\bar{e}(a) + \beta_2(a)\left[\bar{i}_1(a) + \bar{i}_2(a)\right]\right\}\mathrm{d}a = k(a)\bar{V}.$$

由方程组(6)解得

$$\bar{e}(a) = \int_0^a \bar{\lambda}(\xi)\mathrm{e}^{-\int_\xi^a[\phi(\sigma)+\gamma_1(\sigma)]\mathrm{d}\sigma}\mathrm{e}^{-w(a-\xi)}\mathrm{d}\xi,$$

$$\bar{i}_1(a) = \int_0^a \bar{\lambda}(\xi)\mathrm{e}^{-w(a-\xi)}\int_\xi^a \phi(\eta)\mathrm{e}^{-\int_\xi^\eta \phi(\sigma)+\gamma_1(\sigma)\mathrm{d}\sigma}\mathrm{e}^{-\int_\eta^a \varepsilon(\sigma)+\gamma_2(\sigma)\mathrm{d}\sigma}\mathrm{d}\eta\mathrm{d}\xi,$$

$$\bar{i}_2(a) = \int_0^a \bar{\lambda}(\xi)\mathrm{e}^{-w(a-\xi)}\int_\xi^a \varepsilon(\tau)\mathrm{e}^{-\int_\tau^a \gamma_3(\sigma)\mathrm{d}\sigma}\int_\xi^\tau \phi(\eta)\mathrm{e}^{-\int_\xi^\eta \phi(\sigma)+\gamma_1(\sigma)\mathrm{d}\sigma}\mathrm{e}^{-\int_\eta^\tau \varepsilon(\sigma)+\gamma_2(\sigma)\mathrm{d}\sigma}\mathrm{d}\eta\mathrm{d}\tau\,\mathrm{d}\xi.$$

将$\bar{e}(a)$,$\bar{i}_1(a)$, $\bar{i}_2(a)$代入$\bar{V}$得

$$\begin{aligned}1 = \int_0^{a^+} P_\infty(a)\Bigg\{\int_0^a k(\xi)\mathrm{e}^{-w(a-\xi)}\Bigg[&\beta_1(a)\mathrm{e}^{-\int_\xi^a[\phi(\sigma)+\gamma_1(\sigma)]\mathrm{d}\sigma} + \beta_2(a)\int_\xi^a \phi(\eta)\mathrm{e}^{-\int_\xi^\eta \phi(\sigma)+\gamma_1(\sigma)\mathrm{d}\sigma}\mathrm{e}^{-\int_\eta^a \varepsilon(\sigma)+\gamma_2(\sigma)\mathrm{d}\sigma}\mathrm{d}\eta \\ &+ \beta_2(a)\int_\xi^a \varepsilon(\tau)\mathrm{e}^{-\int_\tau^a \gamma_3(\sigma)\mathrm{d}\sigma}\int_\xi^\tau \phi(\eta)\mathrm{e}^{-\int_\xi^\eta \phi(\sigma)+\gamma_1(\sigma)\mathrm{d}\sigma}\mathrm{e}^{-\int_\eta^\tau \varepsilon(\sigma)+\gamma_2(\sigma)\mathrm{d}\sigma}\mathrm{d}\eta\mathrm{d}\tau\Bigg]\mathrm{d}\xi\Bigg\}\mathrm{d}a =: G(w).\end{aligned} \tag{7}$$

定义基本再生数$R_0 = G(0)$,即

$$\begin{aligned}R_0 = \int_0^{a^+} P_\infty(a)\Bigg\{\int_0^a k(\xi)\Bigg[&\beta_1(a)\mathrm{e}^{-\int_\xi^a \phi(\sigma)+\gamma_1(\sigma)\mathrm{d}\sigma} + \beta_2(a)\int_\xi^a \phi(\eta)\mathrm{e}^{-\int_\xi^\eta \phi(\sigma)+\gamma_1(\sigma)\mathrm{d}\sigma}\mathrm{e}^{-\int_\eta^a \varepsilon(\sigma)+\gamma_2(\sigma)\mathrm{d}\sigma}\mathrm{d}\eta \\ &+ \beta_2(a)\int_\xi^a \varepsilon(\tau)\mathrm{e}^{-\int_\tau^a \gamma_3(\sigma)\mathrm{d}\sigma}\int_\xi^\tau \phi(\eta)\mathrm{e}^{-\int_\xi^\eta \phi(\sigma)+\gamma_1(\sigma)\mathrm{d}\sigma}\mathrm{e}^{-\int_\eta^\tau \varepsilon(\sigma)+\gamma_2(\sigma)\mathrm{d}\sigma}\mathrm{d}\eta\mathrm{d}\tau\Bigg]\mathrm{d}\xi\Bigg\}\mathrm{d}a.\end{aligned}$$

3.2　无病平衡点的稳定性

定理 2　若$R_0 < 1$,则无病平衡点$E_0(0, 1, 0, 0, 0, 0)$是局部渐近稳定的;若$R_0 > 1$,则无病平衡点$E_0(0, 1, 0, 0, 0, 0)$不稳定.

证明　当w为实数时,

$$G'(w) < 0, \quad \lim_{w \to +\infty} G(w) = 0, \quad \lim_{w \to -\infty} G(w) = +\infty.$$

当$G(0) < 1$时,方程(7)有唯一的负实根w^*,w^*是$G(w) = 1$的占优实根,此时无病平衡点E_0局部渐近稳定,设$w = x + \mathrm{i}y$是方程(7)的任意复数根,由$1 = G(w^*) = \left|G(x + \mathrm{i}y)\right| \leqslant G(x)$和$G(w)$关于$w$的单调递减性知$\mathrm{Re}\,w < w^*$,故$w^*$是$G(w) = 1$的占优实根;当$G(0) > 1$时,方程(7)有唯一的正实根,此时无病平衡点$E_0$不稳定. 由此可知当$R_0 < 1$时,无病平衡点$E_0$局部渐近稳定;当$R_0 > 1$时,无病平衡点$E_0$不稳定.

定理 3　若$R_0 < 1$,则无病平衡点$E_0(0, 1, 0, 0, 0, 0)$全局渐近稳定.

证明　对模型(4)沿特征线积分得

$$m(a,t) = m(0,t-a)\mathrm{e}^{-\int_0^a \delta(\sigma)\mathrm{d}\sigma}, \quad s(a,t) = \mathrm{e}^{-\int_0^a[k(\sigma)V(\sigma-a+t)]\mathrm{d}\sigma} + \int_0^a \delta(\xi)m(\xi,\xi-a+t)\mathrm{e}^{-\int_\xi^a[k(\sigma)V(\sigma-a+t)]\mathrm{d}\sigma}\mathrm{d}\xi,$$

$$e(a,t) = \int_0^a k(\xi)\bar{V}(\xi-a+t)s(\xi,\xi-a+t)\mathrm{e}^{-\int_\xi^a[\phi(\sigma)+\gamma_1(\sigma)]\mathrm{d}\sigma}\mathrm{d}\xi,$$

$$i_1(a,t) = \int_0^a k(\xi)\bar{V}(\xi-a+t)s(\xi,\xi-a+t)\int_\xi^a \phi(\eta)\mathrm{e}^{-\int_\xi^\eta[\phi(\sigma)+\gamma_1(\sigma)]\mathrm{d}\sigma}\mathrm{e}^{-\int_\eta^a[\varepsilon(\sigma)+\gamma_2(\sigma)]\mathrm{d}\sigma}\mathrm{d}\eta\mathrm{d}\xi,$$

$$i_2(a,t) = \int_0^a k(\xi)\bar{V}(\xi-a+t)s(\xi,\xi-a+t)\int_\xi^a \varepsilon(\tau)\mathrm{e}^{-\int_\tau^a \gamma_3(\sigma)\mathrm{d}\sigma}\int_\xi^\tau \phi(\eta)\mathrm{e}^{-\int_\xi^\eta[\phi(\sigma)+\gamma_1(\sigma)]\mathrm{d}\sigma}\mathrm{e}^{-\int_\eta^\tau[\varepsilon(\sigma)+\gamma_2(\sigma)]\mathrm{d}\sigma}\mathrm{d}\eta\mathrm{d}\tau\mathrm{d}\xi,$$

其中$a < t$. 将$e(a,t)$,$i_1(a,t)$,$i_2(a,t)$代入$\bar{V}(t)$得

$$\begin{aligned}\bar{V}(t) = \int_0^t P_\infty(a)\Bigg\{\int_0^a k(\xi)\bar{V}(\xi-a+t)s(\xi,\xi-a+t) \\ \cdot\Bigg[\beta_1(a)\mathrm{e}^{-\int_\xi^a[\phi(\sigma)+\gamma_1(\sigma)]\mathrm{d}\sigma} + \beta_2(a)\int_\xi^a \phi(\eta)\mathrm{e}^{-\int_\xi^\eta[\phi(\sigma)+\gamma_1(\sigma)]\mathrm{d}\sigma}\mathrm{e}^{-\int_\eta^a[\varepsilon(\sigma)+\gamma_2(\sigma)]\mathrm{d}\sigma}\mathrm{d}\eta\end{aligned}$$

$$+\beta_2(a)\int_{\xi}^{a}\varepsilon(\tau)\mathrm{e}^{-\int_{\tau}^{a}\gamma_3(\sigma)\mathrm{d}\sigma}\int_{\xi}^{\tau}\phi(\eta)\mathrm{e}^{-\int_{\xi}^{\eta}[\phi(\sigma)+\gamma_1(\sigma)]\mathrm{d}\sigma}\mathrm{e}^{-\int_{\eta}^{\tau}[\varepsilon(\sigma)+\gamma_2(\sigma)]\mathrm{d}\sigma}\mathrm{d}\eta\,\mathrm{d}\tau\Bigg]\mathrm{d}\xi\Bigg\}\mathrm{d}a$$
$$+\int_{t}^{a^{+}}P_{\infty}(a)\Big[\beta_1(a)e(a,t)+\beta_2(a)\big(i_1(a,t)+i_2(a,t)\big)\Big]\mathrm{d}a. \tag{8}$$

当$t\to+\infty$时，$\int_{t}^{a^{+}}P_{\infty}(a)\Big[\beta_1(a)e(a,t)+\beta_2(a)\big(i_1(a,t)+i_2(a,t)\big)\Big]\mathrm{d}a\to 0.$

又因$s(a,t)\leqslant 1$，所以对式(8)两端取极限，由Fatou引理（张建平等，2014）知

$$\limsup_{t\to+\infty}\bar{V}(t)\leqslant R_0\limsup_{t\to+\infty}\bar{V}(t).$$

当$R_0<1$时，$\bar{V}(t)=0$. 从而得

$$\lim_{t\to+\infty}e(a,t)=0,\quad \lim_{t\to+\infty}i_1(a,t)=0,\quad \lim_{t\to+\infty}i_2(a,t)=0,\quad \lim_{t\to+\infty}r(a,t)=0.$$

利用特征线积分得$\lim\limits_{t\to+\infty}s(a,t)=1$，$\lim\limits_{t\to+\infty}m(a,t)=0$. 故当$R_0<1$时，无病平衡点$E_0$全局渐近稳定.

4 地方病平衡点的存在性

模型(4)的地方病平衡点$E^*\big(m^*(a),\ s^*(a),e^*(a),i_1^*(a),i_2^*(a),r^*(a)\big)$满足系统

$$\begin{cases}\dfrac{\mathrm{d}m^*(a)}{\mathrm{d}t}=-\delta(a)m^*(a),\\ \dfrac{\mathrm{d}s^*(a)}{\mathrm{d}t}=\delta(a)m^*(a)-\lambda^*(a)s^*(a),\\ \dfrac{\mathrm{d}e^*(a)}{\mathrm{d}t}=\lambda^*(a)s^*(a)-[\phi(a)+\gamma_1(a)]e^*(a),\\ \dfrac{\mathrm{d}i_1{}^*(a)}{\mathrm{d}t}=\phi(a)e^*(a)-[\varepsilon(a)+\gamma_2(a)]i_1{}^*(a),\\ \dfrac{\mathrm{d}i_2{}^*(a)}{\mathrm{d}t}=\varepsilon(a)i_1{}^*(a)-\gamma_3(a)i_2{}^*(a),\\ \dfrac{\mathrm{d}r^*(a)}{\mathrm{d}t}=\gamma_1(a)e^*(a)+\gamma_2(a)i_1{}^*(a)+\gamma_3(a)i_2{}^*(a),\\ m^*(0)=\int_0^{a^+}b(a)\big[m^*(a)+(1-k_1)e^*(a)+(1-k_2)i_1{}^*(a)+(1-k_3)i_2{}^*(a)+r^*(a)\big]\mathrm{e}^{-\int_0^a\mu(\sigma)\mathrm{d}\sigma}\mathrm{d}a,\\ s^*(0)=\int_0^{a^+}b(a)s^*(a)\mathrm{e}^{-\int_0^a\mu(\sigma)\mathrm{d}\sigma}\mathrm{d}a,\\ e^*(0)=\int_0^{a^+}b(a)\{k_1e^*(a)+k_2i_1{}^*(a)+k_3i_2{}^*(a)\}\mathrm{e}^{-\int_0^a\mu(\sigma)\mathrm{d}\sigma}\mathrm{d}a,\\ i_1{}^*(0)=i_2{}^*(0)=r^*(0)=0,\\ m^*(a)+s^*(a)+e^*(a)+i_1{}^*(a)+i_2{}^*(a)+r^*(a)=1,\\ \lambda^*(a)=k(a)\int_0^{a^+}P_{\infty}(a)\big\{\beta_1(a)e^*(a)+\beta_2(a)[i_1{}^*(a)+i_2{}^*(a)]\big\}\mathrm{d}a=k(a)V^*.\end{cases} \tag{9}$$

由系统(9)解得

$$m^*(a)=m^*(0)\mathrm{e}^{-\int_0^a\delta(\sigma)\mathrm{d}\sigma},\qquad s^*(a)=s^*(0)\mathrm{e}^{-\int_0^a k(\sigma)V^*\mathrm{d}\sigma}+\int_0^a\delta(\zeta)m^*(0)\mathrm{e}^{-\int_0^{\zeta}\delta(\sigma)\mathrm{d}\sigma}\mathrm{e}^{-\int_{\zeta}^{a}k(\sigma)V^*\mathrm{d}\sigma}\mathrm{d}\zeta,$$
$$e^*(a)=e^*(0)A_1+V^*(A_2+A_3),\qquad i_1{}^*(a)=e^*(0)B_1+V^*(B_2+B_3),\qquad i_2{}^*(a)=e^*(0)C_1+V^*(C_2+C_3),$$

其中

$$A_1=\mathrm{e}^{-\int_0^a[\phi(\sigma)+\gamma_1(\sigma)]\mathrm{d}\sigma},\qquad A_2=\int_0^a k(\xi)s^*(0)\mathrm{e}^{-\int_{\xi}^{a}[\phi(\sigma)+\gamma_1(\sigma)]\mathrm{d}\sigma}\mathrm{e}^{-\int_0^{\xi}k(\sigma)V^*\mathrm{d}\sigma}\mathrm{d}\xi,$$
$$A_3=\int_0^a k(\xi)\mathrm{e}^{-\int_{\xi}^{a}[\phi(\sigma)+\gamma_1(\sigma)]\mathrm{d}\sigma}\int_0^{\xi}\delta(\zeta)m^*(0)\mathrm{e}^{-\int_0^{\zeta}\delta(\sigma)\mathrm{d}\sigma}\mathrm{e}^{-\int_{\zeta}^{\xi}k(\sigma)V^*\mathrm{d}\sigma}\mathrm{d}\zeta\mathrm{d}\xi,\quad B_1=\int_0^a\phi(\eta)\mathrm{e}^{-\int_{\eta}^{a}[\varepsilon(\sigma)+\gamma_2(\sigma)]\mathrm{d}\sigma}\mathrm{e}^{-\int_0^{\eta}[\phi(\sigma)+\gamma_1(\sigma)]\mathrm{d}\sigma}\mathrm{d}\eta,$$
$$B_2=\int_0^a k(\xi)s^*(0)\mathrm{e}^{-\int_0^{\xi}k(\sigma)V^*\mathrm{d}\sigma}\int_{\xi}^{a}\phi(\eta)\mathrm{e}^{-\int_{\eta}^{a}[\varepsilon(\sigma)+\gamma_2(\sigma)]\mathrm{d}\sigma}\mathrm{e}^{-\int_{\xi}^{\eta}[\phi(\sigma)+\gamma_1(\sigma)]\mathrm{d}\sigma}\mathrm{d}\,\eta\mathrm{d}\xi,$$

$$B_3=\int_0^a\phi(\eta)\mathrm{e}^{-\int_\eta^a[\varepsilon(\sigma)+\gamma_2(\sigma)]\mathrm{d}\sigma}\int_0^\eta k(\xi)\mathrm{e}^{-\int_\xi^\eta[\phi(\sigma)+\gamma_1(\sigma)]\mathrm{d}\sigma}\int_0^\xi\delta(\zeta)m^*(0)\mathrm{e}^{-\int_0^\zeta\delta(\sigma)\mathrm{d}\sigma}\mathrm{e}^{-\int_\zeta^\xi k(\sigma)V^*\mathrm{d}\sigma}\mathrm{d}\zeta\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$C_1=\int_0^a\varepsilon(\tau)\mathrm{e}^{-\int_\tau^a\gamma_3(\sigma)\mathrm{d}\sigma}\int_0^\tau\phi(\eta)\mathrm{e}^{-\int_\eta^\tau[\varepsilon(\sigma)+\gamma_2(\sigma)]\mathrm{d}\sigma}\mathrm{e}^{-\int_0^\eta[\phi(\sigma)+\gamma_1(\sigma)]\mathrm{d}\sigma}\mathrm{d}\eta\mathrm{d}\tau,$$

$$C_2=\int_0^a k(\xi)s^*(0)\mathrm{e}^{-\int_0^\xi k(\sigma)V^*\mathrm{d}\sigma}\int_\xi^a\varepsilon(\tau)\mathrm{e}^{-\int_\tau^a\gamma_3(\sigma)\mathrm{d}\sigma}\int_\xi^\tau\phi(\eta)\mathrm{e}^{-\int_\eta^\tau[\varepsilon(\sigma)+\gamma_2(\sigma)]\mathrm{d}\sigma}\mathrm{e}^{-\int_\xi^\tau[\phi(\sigma)+\gamma_1(\sigma)]\mathrm{d}\sigma}\mathrm{d}\eta\mathrm{d}\tau\mathrm{d}\xi,$$

$$C_3=\int_0^a\varepsilon(\tau)\mathrm{e}^{-\int_\tau^a\gamma_3(\sigma)\mathrm{d}\sigma}\int_0^\tau\phi(\eta)\mathrm{e}^{-\int_\eta^\tau[\varepsilon(\sigma)+\gamma_2(\sigma)]\mathrm{d}\sigma}\int_0^\eta k(\xi)\mathrm{e}^{-\int_\xi^\eta[\phi(\sigma)+\gamma_1(\sigma)]\mathrm{d}\sigma}\int_0^\xi\delta(\zeta)m^*(0)\mathrm{e}^{-\int_0^\zeta\delta(\sigma)\mathrm{d}\sigma}\mathrm{e}^{-\int_\zeta^\xi k(\sigma)V^*\mathrm{d}\sigma}\mathrm{d}\zeta\mathrm{d}\xi\,\mathrm{d}\eta\mathrm{d}\tau,$$

$$s^*(0)=\frac{m^*(0)F_2(V^*)}{1-F_1(V^*)},\quad e^*(0)=\frac{V^*F_4(V^*)}{1-F_3(V^*)},\quad F_1(V^*)=\int_0^{a^+}b(a)\mathrm{e}^{-\int_0^a\mu(\sigma)\mathrm{d}\sigma}\mathrm{e}^{-\int_0^a k(\sigma)V^*\mathrm{d}\sigma}\mathrm{d}a,$$

$$F_2(V^*)=\int_0^{a^+}\int_0^a b(a)\mathrm{e}^{-\int_0^a\mu(\sigma)\mathrm{d}\sigma}\delta(\xi)\mathrm{e}^{\int_0^\xi\delta(\sigma)\mathrm{d}\sigma}\mathrm{e}^{-\int_\xi^a k(\sigma)V^*{}_0\mathrm{d}\sigma}\mathrm{d}\xi\,\mathrm{d}a,\quad F_3(V^*)=\int_0^{a^+}b(a)\mathrm{e}^{-\int_0^a\mu(\sigma)\mathrm{d}\sigma}[k_1A_1+k_2B_1+k_3C_1]\mathrm{d}a,$$

$$F_4(V^*)=\int_0^{a^+}b(a)\mathrm{e}^{-\int_0^a\mu(\sigma)\mathrm{d}\sigma}[k_1(A_2+A_3)+k_2(B_2+B_3)+k_3(C_2+C_3)]\mathrm{d}a.$$

将$e^*(a), i_1{}^*(a), i_2{}^*(a)$代入$V^*$并化简得

$$\begin{aligned}1=\int_0^{a^+}P_\infty(a)\Bigg\{&\frac{F_4(V^*)}{1-F_3(V^*)}\big(\beta_1(a)A_1+\beta_2(a)B_1+\beta_2(a)C_1\big)\\&+\big[\beta_1(a)(A_2+A_3)+\beta_2(a)(B_2+B_3+C_2+C_3)\big]\Bigg\}\mathrm{d}a=:H(V^*).\end{aligned}$$

令$\beta(a)=\max\{\beta_1(a),\beta_2(a)\}$.因为$m^*(a)+s^*(a)+e^*(a)+i_1{}^*(a)+i_2{}^*(a)+r^*(a)=1$,所以对任意的$V^*>0$,有

$$H(V^*)=\frac{1}{V^*}\int_0^{a^+}P_\infty(a)\{\beta_1(a)e^*(a)+\beta_2(a)[i_1{}^*(a)+i_2{}^*(a)]\}\mathrm{d}a\leqslant\frac{\|\beta\|_{L^\infty(0,a^+)}}{V^*}\int_0^{a^+}P_\infty(a)\mathrm{d}a=\frac{\|\beta\|_{L^\infty(0,a^+)}P}{V^*}.$$

记$V^*=\|\beta\|_{L^\infty(0,a^+)}P$,则$H\left(\|\beta\|_{L^\infty(0,a^+)}P\right)<1$,即$H(V^*)<1$.

$$\begin{aligned}H(0)=\int_0^{a^*}P_\infty(a)\Bigg\{&\frac{F_4(0)}{1-F_3(0)}\big(\beta_1(a)A_1+\beta_2(a)B_1+\beta_2(a)C_1\big)\\&+\big[\beta_1(a)(A_2+A_3)+\beta_2(a)(B_2+B_3+C_2+C_3)\big]\Bigg\}\mathrm{d}a\\\geqslant\int_0^{a^+}P_\infty(a)\Bigg\{&\int_0^a k(\xi)\Bigg[\beta_1(a)\mathrm{e}^{-\int_\xi^a[\varepsilon(\sigma)+\gamma_1(\sigma)]\mathrm{d}\sigma}+\beta_2(a)\int_\xi^a\phi(\eta)\mathrm{e}^{-\int_\xi^\eta\phi(\sigma)+\gamma_1(\sigma)\mathrm{d}\sigma}\mathrm{e}^{-\int_\eta^a\varepsilon(\sigma)+\gamma_2(\sigma)\mathrm{d}\sigma}\mathrm{d}\eta\\&+\beta_2(a)\int_\xi^a\varepsilon(\tau)\mathrm{e}^{-\int_\tau^a\gamma_3(\sigma)\mathrm{d}\sigma}\int_\xi^\tau\phi(\eta)\mathrm{e}^{-\int_\xi^\eta\phi(\sigma)+\gamma_1(\sigma)\mathrm{d}\sigma}\mathrm{e}^{-\int_\eta^\tau\varepsilon(\sigma)+\gamma_2(\sigma)\mathrm{d}\sigma}\mathrm{d}\eta\mathrm{d}\tau\Bigg]\mathrm{d}\xi\Bigg\}\mathrm{d}a=R_0>1.\end{aligned}$$

由于$H(V^*)$是关于V^*单调递减的连续函数，且$H(0)>R_0$，若$R_0>1$，则$H(0)>1$. 故$H(V^*)=1$在$\left(0,\|\beta\|_{L^\infty(0,a^+)}P\right)$上存在唯一的正解.

定理4　当$R_0>1$时,模型(4)存在唯一的地方病平衡点$E^*(m^*(a),s^*(a),e^*(a),i_1^*(a),i_2^*(a),r^*(a))$.

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