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半线性退化椭圆方程组解的奇异性和退化估计*

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摘　要： 基于 Re-scaling 变换及 Double 引理，建立半线性退化椭圆方程组解的奇异性和退化估计。作为应用，在有界区域上，证明带有边值问题退化椭圆方程组正解的先验估计。

关键词： Re-scaling 变换；奇异性和退化性；先验估计

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Singularity and decay estimate of solutions of a system of semi-linear degenerate elliptic equations

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Abstract: Some singularity and decay estimates of solutions for a degenerate semi-linear elliptic equation system are established based on re-scaling arguments combined with a doubling property. As an application, a priori bound of solutions of a boundary value problem to a degenerate semi-linear elliptic equation system is derived.

Key words: re-scaling; singularity and decay; a priori bound

本文研究如下半线性退化椭圆方程组

$$\begin{cases}-\mathrm{div}(|x|^{\theta_1}\nabla u) = |x|^{l_1} f(v), & x \in \Omega, \\ -\mathrm{div}(|x|^{\theta_2}\nabla v) = |x|^{l_2} g(u), & x \in \Omega,\end{cases} \tag{1}$$

其中，θ_1, θ_2, l_1, $l_2 \in \mathbb{R}$，$\Omega \subset \mathbb{R}^N (N \geqslant 2)$。函数 $f, g: [0, \infty) \to \mathbb{R}$ 连续。令

$$\tau_i = l_i - \theta_i > -2, \tag{2}$$

$$p_c = \frac{N+2}{N-2},\quad p^i_c := \frac{N_i' + 2 + 2\tau_i}{N_i' - 2}\ (>1),\quad N_i' = N + \theta_i > 0,\ i = 1, 2.$$

当 $\theta_1 = \theta_2 = l_1 = l_2 = 0, f(v) = v^p, g(u) = u^q$ 时，方程组（1）是 Lane-Emden 方程组。著名的 Lane-Emden 猜想是指：在 $\Omega = \mathbb{R}^N$ 中，方程组（1）没有正解当且仅当 $\frac{N}{p+1} + \frac{N}{q+1} > N - 2$。对于径向对称解，猜想在文献［1］中被完全证明。对于非径向对称解，当 $N \leqslant 2$ 时，猜想已由 Mitidieri 等[2] 证明。Polacik 等[3] 和 Souplet[4] 分别证明当 $N = 3$ 和 $N = 4$ 时猜想成立。当 $N \geqslant 5$ 时，Souplet[4] 证明了在条件 $\max\left(\frac{2(p+1)}{pq-1} + \frac{2(q+1)}{pq-1}\right) > N - 3$ 下，猜想也成立。Felmer 等[5] 证明当 $0 < p$, $q \leqslant \frac{N+2}{N-2}$,

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$(p,q) \neq \left(\frac{N+2}{N-2}, \frac{N+2}{N-2}\right)$时，方程组（1）没有$C^2$解。

当$\theta_1 = \theta_2 = 0,\ l_1, l_2 \neq 0,\ f(v) = v^p, g(u) = u^q$时，Bidaut-Veron 等[6]证明了方程组（1）在$\mathbb{R}^N$中没有径向对称解当且仅当$\frac{N+l_1}{p+1} + \frac{N+l_2}{q+1} > N-2$. 对于非径向对称解，当$N=3$和$N=4$，Phan[7]证明了$l_1, l_2 \leqslant 0$时，方程组（1）不存在有界解。

近年来，退化椭圆方程

$$-\mathrm{div}(|x|^\theta \nabla u) = |x|^l u^p, \quad x \in \Omega, \tag{3}$$

引起了广泛的关注。方程（3）作为描述各向异性介质平衡相关物理现象的模型被引入。郭和万等[8]证明了$\Omega = \mathbb{R}^N$中非负解的不存在性。综上，我们进一步研究了方程组（1）。设函数f, g在Ω上连续，$p, q > 1$，且存在常数h, γ使得

$$\lim_{s \to +\infty} \frac{f(s)}{s^p} = h \in (0, +\infty), \quad \lim_{t \to +\infty} \frac{g(t)}{t^q} = \gamma \in (0, +\infty). \tag{4}$$

记

$$\alpha = \frac{2(p+1)}{pq-1}, \quad \beta = \frac{2(q+1)}{pq-1}.$$

1　奇异退化估计

首先，讨论方程组（1）解的奇异性和退化估计，结论如下。

定理 1（奇异退化估计）设$N \geqslant 2$，式（2）和式（4）成立。假设$0 < p, q \leqslant \frac{N+2}{N-2}$，$(p,q) \neq \left(\frac{N+2}{N-2}, \frac{N+2}{N-2}\right)$，则存在常数$C = C(N, p, q, \tau_1, \tau_2) > 0$，使得

（i）在$\Omega = \{x \in \mathbb{R}^N: 0 < |x| < \rho\}(\rho > 0)$中，方程组（1）的任意正解$u, v \in C^2(\Omega \backslash \{0\}) \cap C(\bar{\Omega})$满足

$$\begin{aligned} &u(x) \leqslant C|x|^{-\alpha - \frac{\tau_1 + \tau_2 p}{pq-1}}, \quad v(x) \leqslant C|x|^{-\beta - \frac{\tau_2 + \tau_1 q}{pq-1}}, \quad 0 < |x| < \frac{\rho}{2}, \\ &|\nabla u| \leqslant C|x|^{-\alpha - 1 - \frac{\tau_1 + \tau_2 p}{pq-1}}, \quad |\nabla v| \leqslant C|x|^{-\beta - 1 - \frac{\tau_2 + \tau_1 q}{pq-1}}, \quad 0 < |x| < \frac{\rho}{2}. \end{aligned} \tag{5}$$

（ii）在$\Omega = \{x \in \mathbb{R}^N: |x| > \rho\}(\rho > 0)$中，方程组（1）的任意正解$u, v \in C^2(\Omega \backslash \{0\}) \cap C(\bar{\Omega})$满足

$$\begin{aligned} &u(x) \leqslant C|x|^{-\alpha - \frac{\tau_1 + \tau_2 p}{pq-1}}, \quad v(x) \leqslant C|x|^{-\beta - \frac{\tau_2 + \tau_1 q}{pq-1}}, \quad |x| > 2\rho, \\ &|\nabla u| \leqslant C|x|^{-\alpha - 1 - \frac{\tau_1 + \tau_2 p}{pq-1}}, \quad |\nabla v| \leqslant C|x|^{-\beta - 1 - \frac{\tau_2 + \tau_1 q}{pq-1}}, \quad |x| > 2\rho. \end{aligned} \tag{6}$$

证明　设$\Omega = \{x \in \mathbb{R}^N: 0 < |x| < \rho\}$，$0 < |x_0| < \rho/2$，或者$\Omega = \{x \in \mathbb{R}^N: |x| > \rho\}$，$|x_0| > 2\rho$. 设$R_0 = \frac{|x_0|}{2}$，注意到$\frac{|x_0|}{2} < |x_0 + R_0 y| < \frac{|3x_0|}{2}, y \in B_1(0)$，则$x_0 + R_0 y \in \Omega$. 定义

$$U(y) = R_0^{\alpha + \frac{\tau_1 + \tau_2 p}{pq-1}} u(x_0 + R_0 y), \quad V(y) = R_0^{\beta + \frac{\tau_2 + \tau_1 q}{pq-1}} v(x_0 + R_0 y).$$

则(U, V)满足方程组

$$\begin{cases} -\mathrm{div}\left(\left|y + \frac{x_0}{R_0}\right|^{\theta_1} \nabla U\right) = \left|y + \frac{x_0}{R_0}\right|^{l_1} R_0^{\alpha + \frac{\tau_1 + \tau_2 p}{pq-1} + \tau_1 + 2} f\left(R_0^{-\beta - \frac{\tau_2 + \tau_1 q}{pq-1}} V\right), & y \in B_1, \\ -\mathrm{div}\left(\left|y + \frac{x_0}{R_0}\right|^{\theta_2} \nabla V\right) = \left|y + \frac{x_0}{R_0}\right|^{l_2} R_0^{\beta + \frac{\tau_2 + \tau_1 q}{pq-1} + \tau_2 + 2} g\left(R_0^{-\alpha - \frac{\tau_1 + \tau_2 p}{pq-1}} U\right), & y \in B_1, \end{cases}$$

且有$\left|y+\frac{x_0}{R_0}\right|\in[1,3], y\in\overline{B}_1$.

首先证明存在一个常数$C>0$（与x_0无关），使得

$$\left|U(y)\right|^{\frac{1}{\alpha}}+\left|V(y)\right|^{\frac{1}{\beta}}+\left|\nabla U(y)\right|^{\frac{1}{\alpha+1}}+\left|\nabla V(y)\right|^{\frac{1}{\beta+1}}\leqslant C(1+\operatorname{dist}^{-1}(y,\partial B_1)).$$

反证法。假设存在序列$x_k\in\Omega$及(U_k,V_k)满足

$$\begin{cases}-\operatorname{div}\left(\left|y+\frac{x_k}{R_k}\right|^{\theta_1}\nabla U_k\right)=\left|y+\frac{x_k}{R_k}\right|^{l_1}R_k^{\alpha+\frac{\tau_1+\tau_2 p}{pq-1}+\tau_1+2}f(R_k^{-\beta-\frac{\tau_2+\tau_1 q}{pq-1}}V_k),\quad y\in B_1,\\ -\operatorname{div}\left(\left|y+\frac{x_k}{R_k}\right|^{\theta_2}\nabla V_k\right)=\left|y+\frac{x_k}{R_k}\right|^{l_2}R_k^{\beta+\frac{\tau_2+\tau_1 q}{pq-1}+\tau_2+2}g(R_k^{-\alpha-\frac{\tau_1+\tau_2 p}{pq-1}}U_k),\quad y\in B_1,\end{cases}$$

其中$R_k=\frac{\left|x_k\right|}{2}$，以及存在点列$z_k\in B_1(0)$使得函数

$$M_k=\left|U_k\right|^{\frac{1}{\alpha}}+\left|V_k\right|^{\frac{1}{\beta}}+\left|\nabla U_k\right|^{\frac{1}{\alpha+1}}+\left|\nabla V_k\right|^{\frac{1}{\beta+1}}$$

满足

$$M_k(z_k)>2k(1+\operatorname{dist}^{-1}(z_k,\partial B_1))>2k\operatorname{dist}^{-1}(z_k,\partial B_1).$$

由Double引理[3]可知，存在序列$y_k\in B_1(0)$使得

$$M_k(y_k)>M_k(z_k),\qquad M_k(y_k)>2k\operatorname{dist}^{-1}(y_k,\partial B_1),$$

且

$$M_k(y)\leqslant 2M_k(y_k),\qquad \left|y-y_k\right|\leqslant kM_k^{-1}(y_k).\tag{7}$$

由$M_k(y_k)>M_k(z_k)>2k$可知

$$\lambda_k:=M_k^{-1}(y_k)\to 0,\quad k\to\infty.\tag{8}$$

令

$$\tilde{U}_k(z)=\lambda_k^{\alpha}U_k(y_k+\lambda_k z),\tilde{V}_k(z)=\lambda_k^{\beta}V_k(y_k+\lambda_k z).$$

显然，

$$\left|\tilde{U}_k(0)\right|^{\frac{1}{\alpha}}+\left|\tilde{V}_k(0)\right|^{\frac{1}{\beta}}+\left|\nabla\tilde{U}_k(0)\right|^{\frac{1}{\alpha+1}}+\left|\nabla\tilde{V}_k(0)\right|^{\frac{1}{\beta+1}}=1.\tag{9}$$

由式（7）知

$$\left[\left|\tilde{U}_k\right|^{\frac{1}{\alpha}}+\left|\tilde{V}_k\right|^{\frac{1}{\beta}}\right](z)\leqslant 2,\quad |z|\leqslant k.\tag{10}$$

直接计算可知，$(\tilde{U}_k,\tilde{V}_k)$满足

$$\begin{cases}-\operatorname{div}\left(\left|y_k+\lambda_k z+\frac{x_k}{R_k}\right|^{\theta_1}\nabla\tilde{U}_k\right)=\left|y_k+\lambda_k z+\frac{x_k}{R_k}\right|^{l_1}\underbrace{\lambda_k^{\alpha+2}R_k^{\alpha+\frac{\tau_1+\tau_2 p}{pq-1}+\tau_1+2}f\left(R_k^{-\beta-\frac{\tau_2+\tau_1 q}{pq-1}}\lambda_k^{-\beta}\tilde{V}_k(z)\right)}_{f_k(z)},\quad |z|\leqslant k,\\ -\operatorname{div}\left(\left|y_k+\lambda_k z+\frac{x_k}{R_k}\right|^{\theta_2}\nabla\tilde{V}_k\right)=\left|y_k+\lambda_k z+\frac{x_k}{R_k}\right|^{l_2}\underbrace{\lambda_k^{\beta+2}R_k^{\beta+\frac{\tau_2+\tau_1 q}{pq-1}+\tau_2+2}g\left(R_k^{-\alpha-\frac{\tau_1+\tau_2 p}{pq-1}}\lambda_k^{-\alpha}\tilde{U}_k(z)\right)}_{g_k(z)},\quad |z|\leqslant k.\end{cases}\tag{11}$$

由条件（4）及f,g的连续性可知，$-C\leqslant f(s)\leqslant C(1+s^p), -C\leqslant g(t)\leqslant C(1+t^q), s\geqslant 0, t\geqslant 0,$ 从而

$$-C\lambda_k^{\alpha+2}\leqslant f_k(z)\leqslant C',\ -C\lambda_k^{\beta+2}\leqslant g_k(z)\leqslant C',\ |z|\leqslant k,\ k=1,2,\cdots.$$

注意到对任意的k，有$\left|y_k+\lambda_k z+\frac{x_k}{R_k}\right|\in[1,3]$. 抽取一组子序列（仍记为$\{y_k\},\left\{\frac{x_k}{R_k}\right\}$，$(\tilde{U}_k,\tilde{V}_k)$），由标准嵌入，内部Schauder估计及条件（4）可得，可设$y_k\to y_0\in\bar{B}_1,\frac{x_k}{R_k}\to\tilde{x}\in\partial B_2$及$(\tilde{U}_k,\tilde{V}_k)\to(\tilde{U},\tilde{V})\in C_{loc}^2(\mathbb{R}^N)\times C_{loc}^2(\mathbb{R}^N)$. 再次由条件（4）知，对任意的$z\in\mathbb{R}^N$，当$k\to\infty$时，$f_k(z)\to h\tilde{V}^p$，$g_k(z)\to\gamma\tilde{U}^q$. 因此，$(\tilde{U},\tilde{V})$是

方程

$$\begin{cases}-\Delta\tilde{U} = C\tilde{V}^p, z \in \mathbb{R}^N,\\ -\Delta\tilde{V} = D\tilde{U}^q, z \in \mathbb{R}^N,\end{cases}$$

的非负经典解，其中$0 < C = h|y_0 + \tilde{x}|^{\tau_1} < \infty, 0 < D = \gamma|y_0 + \tilde{x}|^{\tau_2} < \infty$. 且由式（9）可得

$$\left|\tilde{U}(0)\right|^{\frac{1}{\alpha}} + \left|\tilde{V}(0)\right|^{\frac{1}{\beta}} + \left|\nabla\tilde{U}(0)\right|^{\frac{1}{\alpha+1}} + \left|\nabla\tilde{V}(0)\right|^{\frac{1}{\beta+1}} = 1.$$

这与文献［5］中Lane-Emden方程组的Liouville型定理矛盾。这就意味着

$$|\tilde{U}(0)| + |\tilde{V}(0)| + |\nabla\tilde{U}(0)| + |\nabla\tilde{V}(0)| \leqslant C.$$

因此

$$u(x_0) \leqslant C\left|x_0\right|^{-\alpha-\frac{\tau_1+\tau_2 p}{pq-1}},\quad v(x_0) \leqslant C\left|x_0\right|^{-\beta-\frac{\tau_2+\tau_1 q}{pq-1}},$$

且

$$\left|\nabla u(x_0)\right| \leqslant C\left|x_0\right|^{-\alpha-1-\frac{\tau_1+\tau_2 p}{pq-1}},\quad \left|\nabla v(x_0)\right| \leqslant C\left|x_0\right|^{-\beta-1-\frac{\tau_2+\tau_1 q}{pq-1}}.$$

证毕。

定理1的证明是基于Re-scaling变换及Double引理。此外，还需要在全空间中在一定条件下Lane-Emden方程组非平凡解的不存在性。特别地，当$f(v) = v^p, g(u) = u^q$时，方程组（1）变为

$$\begin{cases}-\operatorname{div}(|x|^{\theta_1}\nabla u) = |x|^{l_1}v^p, & x \in \Omega,\\ -\operatorname{div}(|x|^{\theta_2}\nabla v) = |x|^{l_2}u^q, & x \in \Omega.\end{cases} \tag{12}$$

对于方程组（12）也有同样的结论成立，这时将不再需要条件（4）。

2　先验估计

作为定理1的应用，考虑如下边值问题

$$\begin{cases}-\operatorname{div}(|x|^{\theta_1}\nabla u) = |x|^{l_1}f(v), & x \in \Omega,\\ -\operatorname{div}(|x|^{\theta_2}\nabla v) = |x|^{l_2}g(u), & x \in \Omega,\\ (u, v) = (\varphi, \psi), & x \in \partial\Omega,\end{cases} \tag{13}$$

其中$\Omega \subset \mathbb{R}^N$是包含原点的有界光滑区域，$\varphi, \psi \in C(\partial\Omega)$是非负函数。为此，采用Gidas和Spruck在文献［9］中的blow-up方法，有

定理2（先验估计）　设$N \geqslant 2, 1 < p < \min\{p_c, p_c^1\}, 1 < q < \min\{p_c, p_c^2\}, \varphi, \psi \in C(\partial\Omega)$是非负函数，且$\|\varphi\|_{L_\infty}, \|\psi\|_{L_\infty} \leqslant M$. 假设式（2）和式（4）成立，则存在常数$C\left(\Omega, \tau_1, \tau_2, p, q, M\right) > 0$，使得方程组（13）的正解$u, v \in C^2(\Omega\backslash\{0\}) \cap C(\bar{\Omega})$满足

$$\|u\|_{L^\infty(\Omega)} \leqslant C,\qquad \|v\|_{L^\infty(\Omega)} \leqslant C.$$

证明　令$d = \operatorname{dist}(0, \partial\Omega) > 0$. 假设定理2不成立，则由定理1中的估计式（5）可知，除$\{0\} \cup \partial\Omega$外，方程组（13）的所有解都一致有界。因此，只需讨论以下两种可能情形。

情形1　存在一列解(u_k, v_k)及一个点列$P_k \to 0 \in \Omega$，使得

$$N_k = \sup_{|x| < d/2}\left\{u_k^{\frac{1}{\alpha+\frac{\tau_1+\tau_2 p}{pq-1}}}(x) + v_k^{\frac{1}{\beta+\frac{\tau_2+\tau_1 q}{pq-1}}}(x)\right\} = u_k^{\frac{1}{\alpha+\frac{\tau_1+\tau_2 p}{pq-1}}}(P_k) + v_k^{\frac{1}{\beta+\frac{\tau_2+\tau_1 q}{pq-1}}}(P_k) \to \infty,\quad k \to \infty.$$

记

$$\bar{u}(y) = \lambda_k^{\alpha+\frac{\tau_1+\tau_2 p}{pq-1}}u_k(P_k + \lambda_k y),\quad \bar{v}(y) = \lambda_k^{\beta+\frac{\tau_2+\tau_1 q}{pq-1}}v_k(P_k + \lambda_k y),\quad \lambda_k = N_k^{-1}.$$

则$(\bar{u}_k, \bar{v}_k)$满足方程组

$$\begin{cases}-\operatorname{div}\left(\left|y+\dfrac{P_k}{\lambda_k}\right|^{\theta_1}\nabla\bar{u}_k\right)=\left|y+\dfrac{P_k}{\lambda_k}\right|^{l_1}\underbrace{\lambda_k^{\alpha+\frac{\tau_1+\tau_2 p}{pq-1}+\tau_1+2}f\left(\lambda_k^{-\beta-\frac{\tau_2+\tau_1 q}{pq-1}}\bar{v}_k\right)}_{f_k(z)},\quad y\in B_{\frac{d}{2\lambda_k}}(0),\\ -\operatorname{div}\left(\left|y+\dfrac{P_k}{\lambda_k}\right|^{\theta_2}\nabla\bar{v}_k\right)=\left|y+\dfrac{P_k}{\lambda_k}\right|^{l_2}\underbrace{\lambda_k^{\beta+\frac{\tau_2+\tau_1 q}{pq-1}+\tau_2+2}g\left(\lambda_k^{-\alpha-\frac{\tau_1+\tau_2 p}{pq-1}}\bar{u}_k\right)}_{g_k(z)},\quad y\in B_{\frac{d}{2\lambda_k}}(0).\end{cases}\tag{14}$$

同时，从定理1的估计式（5）可知点列$\lambda_k^{-1}|P_k| = |P_k|N_k$是有界的。由此可设$\lambda_k^{-1}P_k\rightarrow\bar{x}, k\rightarrow\infty$. 由条件（4）知，$\|f_k\|_{L^\infty(B_{d/2\lambda_k})},\|g_k\|_{L^\infty(B_{d/2\lambda_k})}$关于$k$有界。由椭圆估计和标准嵌入及条件（4），对于方程组（14），可推得序列$(\bar{u}_k,\bar{v}_k)$收敛$(\bar{u},\bar{v})\in C^2_{loc}(\mathbb{R}^N)\times C^2_{loc}(\mathbb{R}^N)$，$(\bar{u},\bar{v})$满足方程

$$\begin{cases}-\operatorname{div}\left(\left|y+\bar{x}\right|^{\theta_1}\nabla\bar{u}\right)=c\left|y+\bar{x}\right|^{l_1}\bar{v}^p,\quad y\in\mathbb{R}^N,\\ -\operatorname{div}\left(\left|y+\bar{x}\right|^{\theta_2}\nabla\bar{v}\right)=c\left|y+\bar{x}\right|^{l_2}\bar{u}^q,\quad y\in\mathbb{R}^N,\end{cases}$$

且

$$\bar{u}^{\overline{\frac{1}{\alpha+\frac{\tau_1+\tau_2 p}{pq-1}}}}(0)+\bar{v}^{\overline{\frac{1}{\beta+\frac{\tau_2+\tau_1 q}{pq-1}}}}(0)=1.$$

由于$1<p<\min\{p_c,p_c^1\},1<q<\min\{p_c,p_c^2\}$，故此结论与文献［10］中的定理6.2.1矛盾。

情形2　存在一列解(u_k,v_k)及一个点列$P_k\rightarrow P\in\partial\Omega$，使得

$$M_k=\sup_{x\in\Omega:\operatorname{dist}(x,\partial\Omega)<\frac{d}{2}}\left(u_k^{\frac{1}{\alpha}}(x)+v_k^{\frac{1}{\beta}}(x)\right)=u_k^{\frac{1}{\alpha}}(P_k)+v_k^{\frac{1}{\beta}}(P_k)\rightarrow\infty,k\rightarrow\infty.$$

不失一般性，设在点P附近的边界$\partial\Omega$包含在超平面$x_N=0$中。对方程组解做如下变换

$$\bar{u}_k(z)=\lambda_k^{\alpha}u(P_k+\lambda_k z),\ \bar{v}_k(z)=\lambda_k^{\beta}v(P_k+\lambda_k z),\ \lambda_k=M_k^{-1}.$$

设$d_k=\operatorname{dist}(P_k,\partial\Omega)$. 注意到当$k$充分大时，$\bar{u}_k(z),\bar{v}_k(z)$在$B_{\frac{\delta}{2\lambda_k}}(0)\cap\left\{z_k>-\dfrac{d_k}{\lambda_k}\right\}$（$\delta$>0）上有定义且满足方程组

$$\begin{cases}-\operatorname{div}\left(\left|P_k+\lambda_k z\right|^{\theta_1}\nabla\bar{u}_k\right)=\left|P_k+\lambda_k z\right|^{l_1}\lambda_k^{\alpha}f(\lambda_k^{-\beta}\bar{v}_k),\\ -\operatorname{div}\left(\left|P_k+\lambda_k z\right|^{\theta_2}\nabla\bar{v}_k\right)=\left|P_k+\lambda_k z\right|^{l_2}\lambda_k^{\beta}g(\lambda_k^{-\alpha}\bar{u}_k),\end{cases}$$

且$\bar{u}_k^{\frac{1}{\alpha}}(0)+\bar{v}_k^{\frac{1}{\beta}}(0)=1$. 不妨设$\bar{u}_k^{\frac{1}{\alpha}}(0)\geqslant\dfrac{1}{2}$.

首先证明$\dfrac{d_k}{\lambda_k}\geqslant C_1$（$C_1$为正常数）。注意到$\left|\bar{u}_k(0)-\bar{u}_k\left(0',-\dfrac{d_k}{\lambda_k}\right)\right|\leqslant C\cdot\dfrac{d_k}{\lambda_k}$，故

$$\frac{1}{2^{\alpha}}-\lambda_k^{\alpha}\sup\varphi(x)\leqslant C\cdot\frac{d_k}{\lambda_k}.$$

因为$\varphi(x)$有界，当$k\rightarrow\infty$时，$\lambda_k\rightarrow 0$，故可得$\dfrac{d_k}{\lambda_k}$一致下有界，且$\dfrac{d_k}{\lambda_k}$要么无上界，要么存在一子序列，当$k\rightarrow\infty$时，$\dfrac{d_k}{\lambda_k}\rightarrow\kappa(\kappa>0)$. 对于$\dfrac{d_k}{\lambda_k}$无上界的情形，可由情形1的讨论直接得出结论。对于存在一子列$\dfrac{d_k}{\lambda_k}\rightarrow\kappa$的情形，同情形1，抽取一收敛子列（仍记为$(\bar{u}_k,\bar{v}_k)$, 则$(\bar{u}_k,\bar{v}_k)\rightarrow(\bar{u},\bar{v})$，$(\bar{u},\bar{v})$是方程组

$$\begin{cases}-\Delta\bar{u}=c\left|P\right|^{\tau_1}\bar{v}^p,\quad z\in H_\kappa^N,\\ -\Delta\bar{v}=c\left|P\right|^{\tau_2}\bar{u}^q,\quad z\in H_\kappa^N,\\ \bar{u}^{\frac{1}{\alpha}}(0)+\bar{v}^{\frac{1}{\beta}}(0)=1,\quad z\in H_\kappa^N,\\ \bar{u}(0)=v(0)=0,\quad z\in\partial H_\kappa^N,\end{cases}$$

的解，其中$H_\kappa^N=\left\{z\in\mathbb{R}^N:z_N>-\kappa\right\}$. 由于，故此结论与文献［3］定理4.2的结论矛盾。证毕。

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